More exercises on maximal solutions

1 Exercise 1

We consider the differential equation

$$y'(t) = y(t)\sin^2(y(t))$$

1. What are the constant functions that are solutions of this differential equation ?

We assume that $y = C, \in \mathbb{R}$ is a constant solution of the differential equation. Therefore

$$y'(t) = 0 = C\sin^2(C).$$

It means that either $\sin^2(C) = 0 \Leftrightarrow C = k\pi$ with $k \in \mathbb{Z}$, or $C = 0$, which is a particular case of the first possibility.

At the end, the constant solutions of the differential equations are all the functions y, $y = k\pi$ with $k \in \mathbb{Z}$. They are by the way maximal solutions, because they are defined on $\mathbb{R}$.

2. Let y be a maximal solution satisfying $y(0) = y_0$. Prove that y is bounded and monotonous.

Let $k_0 \in \mathbb{Z}$ be such that $k_0\pi \leq y_0 < (k_0+1)\pi$. There are two possibilities :

- $y_0 = k_0\pi$.
  In this case y is a constant solution, thus it is bounded and monotonous,
- $k_0\pi < y_0 < (k_0+1)\pi$.
  Let us assume that y is not bounded. Then there exists a t such that $y(t) = k_0\pi$ or $y(t) = (k_0+1)\pi$. It would mean that the graphs of y and of a maximal constant solution do cross, which is not possible, due to the unicity property. Therefore, y is bounded between $k_0\pi$ and $(k_0+1)\pi$.
  Moreover, we know that
  $$y' = y\underbrace{\sin^2(y)}_{\geq 0}$$
  Therefore y' has the same sign than y. We know that y stays always bounded between $k_0\pi$ and $(k_0+1)\pi$ with $k \in \mathbb{Z}$, therefore y has always the same sign. Consequently, y' is always positive or always negative, which means that y is monotonous.

3. Prove that y is defined over $\mathbb{R}$.

We use the theorem 0.4 of the course (explosion of the solution). The function y is defined over an interval $]a, b[$, with a and b either reals or infinite values. If $b \neq +\infty$, then it means that $y(t)$ tend to $\pm\infty$ when t tend to zero, which is not possible since y is bounded. Therefore, $b = +\infty$. We do the same reasoning to prove that $a = -\infty$. As a conclusion, y is defined over $\mathbb{R}$.

2 Exercise 2

We consider the differential equation

$$y'(t) = \cos(y(t)) + \frac{1}{2}\sin(t)$$

1. With a graphical analysis, find some horizontal barriers. Prove that they are indeed barriers.

On the picture available, you can see a schematic representation of the phase field. Notably, you can observe that the horizontal line $y = \pi$ seems to be a barrier that repells the solutions standing under it, because the slope of the tangent field is always negative on this line.

Similarly, the horizontal line $y = 0$ seems to be another barrier repelling positive solutions above it because the slope of the tangent field is always positive on this line.

We check that the associated constant functions are indeed barriers, that is, sub- or super-solutions of the differential equations :

- If $g_1(t) = \pi \, \forall t \in \mathbb{R}$ then $g_1'(t) = 0 > -\frac{1}{2} > -1 + \frac{1}{2}\sin(t) = \cos(g_1(t)) + \frac{1}{2}\sin(t) \, \forall t \in \mathbb{R}$. Therefore $g(t) = \pi$ is a supersolution of the differential equation.
- If $g_2(t) = 0 \, \forall t \in \mathbb{R}$ then $g_2'(t) = 0 < \frac{1}{2} < 1 + \frac{1}{2}\sin(t) = \cos(g_2(t)) + \frac{1}{2}\sin(t) \, \forall t \in \mathbb{R}$. Therefore $g(t) = 0$ is a subsolution of the differential equation.

2. Let f be the solution satisfying the initial condition $f(0) = 0$. Prove that f is bounded and defined over $\mathbb{R}$.

The solution f has its initial condition equal to the subsolution $g_1(t) = 0$, therefore for all $t > 0$, $f(t) > g_1(t) = 0$.

Similarly, f has its initial condition smaller to the subsolution $g_2(t) = \pi$, therefore for all $t > 0$, $f(t) < g_2(t) = \pi$. Therefore, f is bounded. Using the same reasoning as in the previous exercice, we can conclude that f is defined on $[0, +\infty[$.

The same reasoning about sub- and supersolution can be performed with all constant functions :

$$y(t) = \pi + 2k\pi, \, k \in \mathbb{Z}$$

that are super-solutions, and

$$y(t) = 2k\pi, \, k \in \mathbb{Z}$$

that are sub-solutions.

With the extension theorem, it is then possible to prove that f is defined on $\mathbb{R}$. Let us assume that f cannot be extended to $\mathbb{R}$. It means that there exists $\xi \in \mathbb{R}^-$ such that $|f(t)|$ tends to $+\infty$ when t tends to ξ. But if it is the case, then the graph of f necessarily crosses the graph of one of these sub- or subsolutions, which is not possible. Therefore, f can be extended to $\mathbb{R}$.

3. Prove that every solution is bounded and defined over $\mathbb{R}$.

Every solution, whatever its initial condition, can be bounded by a super-solution and a sub-solution of the form described in the previous question. Therefore, the reasoning of the previous question can be used again.