Epidemiology : modeling of a cold epidemy

This exercise aims at studying the SIS epidemiological model. This model is notably applied for mild diseases as a cold, for which infected individuals very rarely die but can be infected several times. Let us denote N the total population, assumed constant along the time of the study, and $y(t)$ the number of individuals infected at time t. We assume that the variation dy of y during an infinitesimal time dt is caused by the following events :

- the new contaminations, assumed proportional to the product $y(N-y)$, with a constant positive coefficient α,
- the healings, assumed proportional to y, with a constant positive coefficient β.

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1. Explain why such a modelling has been chosen, in particular the term $y(N-y)$.

The number of healings is assumed to be proportional to the number of infected individuals, the number of new contaminations is assumed to be proportional to the number of non-infected people and to the number of infected people : it leads to the term $y(N-y)$. There are just two categories of people because we can consider that no one dies from the cold, and as one person can be infected several times by a cold, there is no need to consider a third category of healed people.

2. Prove that the differential equation corresponding to this model is

$$y'(t) = (\alpha N - \beta)y(t) - \alpha y^2(t).$$

With this model, dy a small variation of y is obtained by :

$$dy = \big(\alpha y(N-y) - \beta y\big)dt$$

it leads to :

$$\frac{dy}{dt} = \alpha y(N-y) - \beta y$$

and we finally obtain the differential equation when dt tends to zero.

3. Solve this equation. You can use the change of variables $u = \frac{1}{y}$. What is the influence of the sign of $\alpha N - \beta$? How does look the graph of the solution in the case $\alpha N - \beta > 0$?

We use the change of variables $u = \frac{1}{y}$, assuming that $y \neq 0$. Therefore

$$u'(t) = -\frac{y'(t)}{y^2(t)}$$
$$y'(t) = -u'(t)\, y^2(t)$$

and inserting this relationship in the differential equation :

$$-u'(t)\, y^2(t) = (\alpha N - \beta)y(t) - \alpha y^2(t)$$
$$-u'(t) = (\alpha N - \beta)u(t) - \alpha$$
$$u'(t) = -(\alpha N - \beta)u(t) + \alpha$$

We recognize a first-order linear differential equation. The solution of this equation is

$$u(t) = Ce^{-(\alpha N-\beta)t} + \frac{\alpha}{\alpha N - \beta}$$

with C a constant that we determine with the initial condition :

$$u(0) = C + \frac{\alpha}{\alpha N - \beta}$$

and we finally obtain

$$u(t) = (u(0) - \frac{\alpha}{\alpha N - \beta})e^{-(\alpha N-\beta)t} + \frac{\alpha}{\alpha N - \beta}$$

and

$$y(t) = \frac{1}{(u(0) - \frac{\alpha}{\alpha N-\beta})e^{-(\alpha N-\beta)t} + \frac{\alpha}{\alpha N-\beta}}$$

If $\alpha N - \beta$ is positive, then u decreases and reaches asymptotically its maximum, the equilibrium $\frac{\alpha}{\alpha N-\beta}$ when t tends to infinity, and thus y increases and tends to the equilibrium $\frac{\alpha N-\beta}{\alpha}$.
If $\alpha N - \beta$ is negative, $u(t)$ increases and tends to infinity and thus $y(t)$ decreases and tends to zero when t tends to infinity.

4. The table hereafter indicates the total number of colds in a small town during December and January, as a function of the number of past days. For instance, on 28th day, there were 1940 colds. Assuming that the SIS model is relevant, and $\alpha = 2.382.10^{-5}$ and $\beta = 0.1572$, use this table to provide an approximate previsional number of the maximum of the total number of colds during one day.

20	22	24	26	28	30	32	34	36	38	40	42	44
480	697	1000	1408	1940	2600	3366	4194	5014	5791	6452	6986	7396

We see that y is increasing, it means that we are in the case where $\alpha N - \beta$ is positive, and thus the maximum of y will be $\frac{\alpha N-\beta}{\alpha}$. We denote in the following $\gamma = \alpha N - \beta$ for the sake of simplicity. We know that

$$u'(t) = -(\alpha N - \beta)u(t) + \alpha$$

Therefore

$$\frac{\alpha - u'(t))}{u(t)} = \gamma$$

We use the table to compute an approximation of $\frac{\alpha - u'(t)}{u(t)}$, see Scilab program attached to the course. Of course it is not compulsary to use Scilab, but with the program we can compute easily approximations of $u'(t)$ for all times and thus check that the obtained value of γ is almost constant over time. With the value of γ we can compute $y_{max} = \frac{\alpha N-\beta}{\alpha}$, and we obtain the approximated value $y_{max} = 8335$.

5. Give an estimation of the number of residents in the town.

Using the value of γ, and knowing α and β we compute an approximated value of N, which is $N \approx 14934$.