Exercises about the geometrical study of ODE's (3) : autonomous equations

1 Use of phase lines and barriers to find the shape of the solutions

We consider the differential equation

$$y'(x) = f(y)(x)$$

with

$$\begin{aligned} f(y) &= y(y-1) && (1) \\ f(y) &= y^3 - 1 && (2) \\ f(y) &= 1 - y^3 && (3) \end{aligned}$$

1. For the phase lines determination, we refer to the additional file.
2. In the case of (2), the unique critical point is $y = 1$. It is an unstable critical point because y' is negative on the left side of this point, and positive on its right side. Therefore, the solution for $y(0) = 2$, which is located "above" the stationary solution $y(x) = 1$, is increasing, because its derivative is always positive.

   Now we aim to prove that this solution is not bounded, but rather explodes, using the function $e^x + 1$ as a subsolution on $[0, +\infty[$. We denote by u the solution of the differential equation such that $u(0) = 2$. We first prove that $g(x) = e^x + 1$ is a subsolution. To this purpose we have to check that for all $x \in \mathbb{R}\, g'(x) < f(x, g(x))$. Here we have

   $$g'(x) = e^x$$

   and

   $$f(x, g(x)) = g(x)^3 - 1 = (e^x + 1)^3 - 1 = e^{3x} + 3e^x + 3e^{2x} > e^x = g'(x)$$

   Moreover, $g(0) = 2 = u(0)$. Therefore, following theorem 0.5 in the course notes, we can deduce that for all $x > 0$, $g(x) < u(x)$.

   We know that $g(x) \to +\infty$ when $x \to +\infty$. Therefore, $u(x)$ explodes. Indeed, either the interval of definition of u includes $+\infty$, and in this case, comparing u and g we know that $u(x) \to +\infty$ when $x \to +\infty$, or the interval of definition of u is bounded on the right side. In which case, u explodes, due to theorem 0.4 in the note courses.
3. Now we are looking for a subsolution g and a supersolution h both tending to 1 when x tends to $+\infty$, with $g(0) = 0$ and $h(0) = 2$. (We want that the super- and subsolution satisfy these initial conditions because the text of the exercise asks to consider solutions of the differential equation with initial conditions between 0 and 2.)

   With inspiration from the last question we can try to use exponential functions, and we can use for instance : $g(x) = 1 - e^{-x}$ and $h(x) = 1 + e^{-x}$.

   $$g'(x) = e^{-x} \text{ and } h'(x) = -e^{-x}$$

$f(x,h(x)) = 1-h(x)^3 = 1-(1+e^{-x})^3 = 1-(1+3e^{-x}+3e^{-2x}+e^{-3x}) = -3e^{-x}-3e^{-2x}-e^{-3x} < -e^{-x} = h'(x)$

$f(x,g(x)) = 1-g(x)^3 = 1-(1-e^{-x})^3 = 1-(1-3e^{-x}+3e^{-2x}-e^{-3x}) = 3e^{-x}-3e^{-2x}+e^{-3x}$

We have to check that $3e^{-x}-3e^{-2x}+e^{-3x} > e^{-x} \iff 2e^{-x}-3e^{-2x}+e^{-3x} > 0$. We denote $Y = e^{-x}$ and we study the sign of $2e^{-x}-3e^{-2x}+e^{-3x} = 2Y-3Y^2+Y^3 = Y(2-3Y+Y^2)$. We have $0 < Y \leq 1$, so we just have to study the sign of $2-3Y+Y^2$.

$$\Delta = 9 - 4 * 2 * 1 = 1$$

The roots are $Y_1 = \frac{3+1}{2} = 2$ and $Y_2 = \frac{3-1}{2} = 1$. The polynomial $2-3Y+Y^2$ is therefore strictly positive if $Y > 2$ or $Y < 1$, which is the case if $x > 0$. Thus, $2Y-3Y^2+Y^3$ is strictly positive, which means that $3e^{-x}-3e^{-2x}+e^{-3x} > e^{-x}$, if $x > 0$. If $x = 0$ then $3e^{-x}-3e^{-2x}+e^{-3x} = e^{-x}$. We conclude that h is a super solution and g almost a sub-solution because for x= 0 the inequality is not strict, but the result of theorem 0.5 in the course notes can still be applied in this case. Therefore, if we consider a solution u of the differential equation with $0 \leq u(0) \leq 2$, from the theorem 0.5 in the course notes, we can conclude that for all $x > 0$ we have :

$$h(x) > u(x) \geq g(x)$$

When x tends to infinity, both $h(x)$ and $g(x)$ tend to 1. Thefore $u(x)$ also tend to 1.

2 Limits of phase lines

We consider the differential equations $x'(t) = x(t)$ and $x'(t) = x^3(t)$. Both differential equations have obviously the same phase line, because they have the same critical point, and the sign of the derivative is the same.

The solutions of the first differential equation are $x(t) = x(0)e^t$, and are defined on $\mathbb{R}$.

On the other side, consider the function $h(t) = (c-t)^{-1/2}$. Its derivative is $h'(t) = \frac{1}{2}(c-t)^{-3/2} < h(t)^3$. Therefore, h is a subsolution for the differential equation $x' = x^3$, on the interval where it is defined.

Let us consider u the solution of the differential equation for the initial condition $x(0) = a \neq 0$ and c such that $h(0) = c^{-1/2} \leq a$. For all $t > 0$ where the functions u and h are defined, $u(t) \geq h(t)$. Moreover, the function $h(t)$ tends to $+\infty$ when t tends to c (by inferior value).

Therefore, u tends to $+\infty$ also, for a value smaller or equal to c. We conclude that u explodes.