Exercises about the geometrical study of ODE's (3) : autonomous equations

1 Use of phase lines and barriers to find the shape of the solutions

We consider the differential equation

$$y'(x) = f(y)(x)$$

with

$$\begin{aligned} f(y) &= y(y-1) && (1) \\ f(y) &= y^3 - 1 && (2) \\ f(y) &= 1 - y^3 && (3) \end{aligned}$$

1. Determine the phase lines for the differential equations corresponding to (1), (2) and (3).

2. In the case of (2), prove that the solution for the initial condition $y(0) = 2$ explodes, using the function $e^x + 1$ as a subsolution on $[0, +\infty[$.

3. In the case of (3), using a supersolution and a subsolution inspired by the last question, prove that all solutions $y(x)$ with initial conditions between 0 and 2 tend to 1 when x tends to ∞.

2 Limits of phase lines

We consider the differential equations $x'(t) = x(t)$ and $x'(t) = x^3(t)$. Check that they have the same phase line, but that the maximal solutions of the first one are defined on $\mathbb{R}$ while the solutions of the second one explode (at the exception of the zero solution).