Exercises about the geometrical study of ODE's (2)

1 Non-explosion of a solution

We consider the differential equation

$$y' = y^2 - x$$

1. We denote $V_{x,y} = y^2 - x$.

$$\begin{aligned} V_{x,y} &= 0 \Leftrightarrow y^2 = x \Leftrightarrow x \geq 0 \text{ and } |y| = \sqrt{x}, \\ V_{x,y} &\leq 0 \Leftrightarrow y^2 \leq x \Leftrightarrow x \geq 0 \text{ and } |y| \leq \sqrt{x}, \\ V_{x,y} &\geq 0 \Leftrightarrow y^2 \geq x \Leftrightarrow (x \geq 0 \text{ and } |y| \geq \sqrt{x}) \text{ or } x \leq 0. \end{aligned}$$

2. Let $M_0 = (x_0, y_0)$ be a point where the slope of the tangent field is negative, and u the solution satisfying $u(x_0) = y_0$. Then following the tangent field, the graph of u will always remain in the part of the plane where the tangent slope is negative, which is bounded for bounded values of x. Therefore, u can not explode.

2 A remark about barriers

We consider a first order ODE written under the following form :

$$y' = f(x, y). \tag{1}$$

Let g be a supersolution, and u a solution of differential equation (1). We assume that there exists x_1 such that $u(x_1) = g(x_1)$.

We make a Taylor series expansion of $g(x) - u(x)$ for x close to x_1 :

$$(g-u)(x) = (g-u)(x_1) + (x - x_1)(g-u)'(x_1) + O((x-x_1)^2) = (x-x_1)(g-u)'(x_1) + O((x-x_1)^2)$$

Therefore, if x is close enough to x_1, then the sign of $(g-u)(x)$ is the same as the sign of $(x-x_1)(g-u)'(x_1)$. Now we remark that

$$(x-x_1)(g-u)'(x_1) = (x-x_1)(g'(x_1) - f(x_1, u(x_1)) = (x-x_1)(g'(x_1) - f(x_1, g(x_1))$$

Because $g'(x_1) - f(x_1, g(x_1) > 0$ we conclude $(g-u)(x)$ has the same sign as $x - x_1$.

3 Barriers and limit of solutions when x tend to $+\infty$

We consider the differential equation

$$y' = -y - \frac{y}{x} \tag{2}$$

for $x \in]0, +\infty[$.

1. We denote $f(x, y) = -y - \frac{y}{x}$.

$$f(x, g(x)) = -g(x)(1 + \frac{1}{x}) < -g(x) = g'(x) \text{ if } g \text{ is positive.}$$

2. Similarly

$$f(x, g(x)) = -g(x)(1 + \frac{1}{x}) > -g(x) = g'(x) \text{ if } g \text{ is negative.}$$

3. The solutions of the differential equation $y' = -y$ can all be written under the form $y(x) = y_0 e^{-(x-x_0)}$, with $y_0 = y(x_0)$.

4. We define by f the maximal solution of the differential equation (2), for the initial condition $f(x_0) = y_0$.
We consider $y_1(x) = -|y_0|e^{-x}$ and $y_2(x) = |y_0|e^{-x}$. The function y_1 are both barriers for the differential equation (2) : y_1 is a subsolution, and y_2 a supersolution. Due to a theorem in the course notes, we deduce that for all x on which f is defined, $y_1(x) \leq f(x) \leq y_2(x)$. Therefore, f does not explode on a finite interval. Consequently, f is defined on $]x_0, +\infty[$. As y_1 and y_2 both tend to 0 when x tends to $+\infty$, we conclude that f also tends to 0 when x tends to $+\infty$.