Exercises about the geometrical study of ODE's (2)

1 Non-explosion of a solution

We consider the differential equation

$$y' = y^2 - x$$

1. Determine the three regions of the (x, y)-plane where the slope of the tangent field is zero, positive, negative.
2. Let $M_0 = (x_0, y_0)$ be a point where the slope of the tangent field is negative, and u the solution satisfying the initial $u(x_0) = y_0$. Prove that u does not explode (that is, does not tend to infinity when x tends to a finite value).

2 A remark about barriers

We consider a first order ODE written under the following form :

$$y' = f(x, y). \tag{1}$$

Let g be a supersolution, and u a solution of differential equation (1). We assume that there exists x_1 such that $u(x_1) = g(x_1)$.

Using a Taylor series expansion of $g - u$, prove that
– if $x < x_1$ and is enough close to x_1 , then $g(x) < u(x)$.
– if $x > x_1$ and is enough close to x_1 , then $g(x) > u(x)$.
The same kind of results can be proven for a subsolution.

3 Barriers and limit of solutions when x tend to $+\infty$

We consider the differential equation

$$y' = -y - \frac{y}{x} \tag{2}$$

for $x \in]0, +\infty[$. We notice that the zero function is solution of this ODE.

1. Prove that, if g is a positive solution for the differential equation $y' = -y$, then the graph of g is a barrier for the differential equation (2).
2. Similarly, prove that, if h is a negative solution for the differential equation $y' = -y$, then the graph of h is a barrier for (2).

3. What are the solutions of the differential equation $y' = -y$?

4. We define by f the maximal solution of the differential equation (2), for the initial condition $f(x_0) = y_0$. Using the first questions, prove that f is defined (at least) on $[x_0, +\infty[$ and that $f(x) \to 0$ when $x \to +\infty$.