Geometrical study of ODE's (2)

(These notes are strongly inspired by the course of Frédéric Le Roux, at University Paris 11)

We consider a first order ODE written under the following form :

$$y' = f(x, y). \tag{1}$$

Theorem 0.1. ***(Extension)***
We assume that f is continuous for $x_1 < x < x_2$ and for all y, and that there exists a function $\phi(x)$ satisfying the following conditions :
- *ϕ and ϕ' are continuous in a subinterval I of $]x_1, x_2[$.*
- *$\phi'(x) = f(x, \phi(x))$ for $x \in I$.*

Then, either
- *$\phi(x)$ can be extended to the entire interval $]x_1, x_2[$ as a solution of the ODE (1), or,*
- *$|\phi(x)| \to +\infty$ when $x \to \xi$ for some $\xi \in]x_1, x_2[$.*

For instance, on Fig. 1 are plotted some vectors $V_{x,y}$ for $f(x, y) = y^2$. Following the tangent lines, we observe that the solutions seem to tend to infinity when x increases. It means that the solution can not be extended to high values of x.

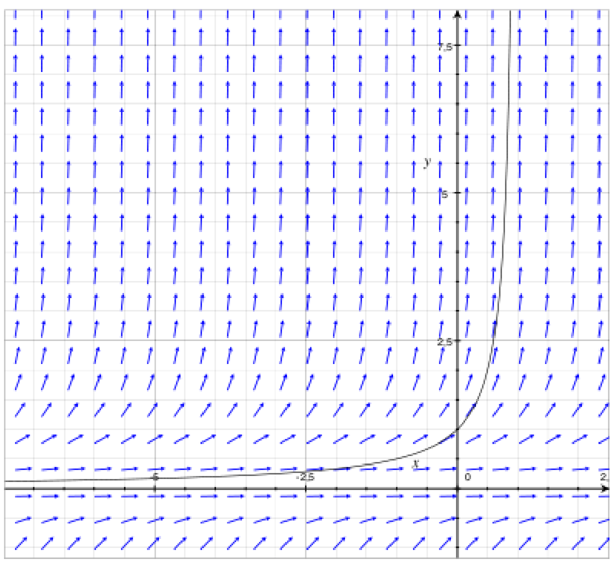

FIGURE 1 – Example of a tangent field corresponding to solutions exploding ($f(x, y) = y^2$)

Generally, when we consider the differential equation (1) with a specific initial condition $v(x_0) = y_0$, we look for the "biggest" interval containing x_0, on which a solution exists. This leads to the following definition :

Definition (maximal solution) :
We consider the ODE (1) with an initial condition $v(x_0) = y_0$ A maximal solution for this problem is a function v defined on an interval I, called life interval, such that :

— f is solution of (1) and satisfies the initial condition

— there exists no solution w of the same equation, satisfying the same initial condition, defined on an interval J containing I and bigger than I.

Theorem 0.2. ***Existence and unicity :***
We consider the ODE (1), and assume that f is defined and continuously derivable for all $x \in I$ and for all $y \in J$ with I and J intervals. Then, for all initial condition $u(x_0) = y_0$ with $x_0 \in I$ and $y_0 \in J$, there exists an unique maximal solution of the differential equation satisfying this initial condition.

Corollary 0.3. *If u_1 and u_2 are solutions of the same differential equation on an interval I, and if there exists a point x_0 such that $u_1(x_0) = u_2(x_0)$ then $u_1 = u_2$ on I.*

With this theorem, we can obtain useful properties of the solution of ODE's. For instance, if we consider $y' = y^2$ with the initial condition $y(0) = 1$, we can prove that for all x where y is defined, $y(x) \geq 0$. Indeed, one solution to this ODE is the zero function $y_1(x) = 0 \, \forall x$. The graphs of y and y_1 can not cross, because of the theorem. Therefore, as $y(0) > 0$ and y is continuous (because its derivative is defined), we deduce that $y(x) \leq 0$ for all x in the interval where y is defined.

Theorem 0.4. ***Explosion of the solution :***
We consider u a maximal solution defined on an interval $I =]a, b[$. If $b \neq +\infty$ then u tends to $+\infty$ or $-\infty$ when x tends to b^-. The conclusion is similar if $a \neq -\infty$

Definition (supersolution, subsolution and barrier) :
A supersolution is a derivable function g such that for all $x \in \mathbb{R}$ $g'(x) > f(x, g(x))$. Similarly, a subsolution is a derivable function g such that for all $x \in \mathbb{R}$ $g'(x) < f(x, g(x))$. The graph of a subsolution or a supersolution is called a barrier for the differential equation that we consider.

For instance, let us consider the differential equation $y' = 2\,cos(y - x)$. The function $g(x) = x + \pi$ is a supersolution for this differential equation.

The interest of barriers is that they are easier to find that solutions, and can provide informations about the behavior of solutions of the differential equation. Let us also remark that super- and sub solutions can also be defined on intervals of the type $[a, +\infty[$ with a a real, instead of $\mathbb{R}$.

Theorem 0.5. *We denote g a supersolution for the differential equation 1, and y a solution of this differential equation. If there exists $x_0 \in \mathbb{R}$ such that $g(x_0) \geq y(x_0)$, then for all $x \geq x_0$ we have $g(x) > y(x)$.*

If, instead of the strict inequality $g'(x) > f(x, g(x))$, we have $g'(x) \geq f(x, g(x))$, then we can conclude under the same hypothesis that for all $x \geq x_0$ we have $g(x) \geq y(x)$.

A similar theorem can be proven for a subsolution.

We consider again the differential equation $y' = 2\,cos(y-x)$. The functions $h(x) = x$ and $g(x) = x+\pi$ are respectively sub- and supersolutions. Let u be a solution of 1 such that $f(0) \in [0, \pi]$. Applying the last theorem, we find that the graph of u is between the graphs of h and g. We deduce that $u(x)/x$ tends to 1 when x tends to ∞.