Exercises on numerical schemes for differential equations

1 Comparison of numerical schemes

We consider the differential equations :

$$x'(t) = x(t)$$

and

$$x'(t) = x(t) + t - 1$$

with the initial condition $x(0) = 1$.

1. For each of these equations, compute the numerical solution at time $t = 0.2$ using the time step $\Delta t = 0.1$, with the Euler method, the Heun's method, the second order Runge-Kutta method.

– Euler method :

(a) For $x'(t) = x(t)$

$$\begin{aligned} x_{n+1} &= x_n + \Delta t\, f(t_n, x_n) \\ x_0 &= x(0) = 1 \\ x_1 &= x_0 + \Delta t f(t_0, x_0) \\ x_1 &= 1. + 0.1 \times 1. = 1.1 \\ x_2 &= x_1 + \Delta t\, f(t_1, x_1) \\ x_2 &= 1.1 + 0.1 \times 1.1 = 1.21 \end{aligned}$$

(b) For $x'(t) = x(t) + t - 1$

$$\begin{aligned} x_{n+1} &= x_n + \Delta t f(t_n, x_n) \\ x_0 &= x(0) = 1 \\ x_1 &= x_0 + \Delta t f(t_0, x_0) \\ x_1 &= 1. + 0.1 \times (1. + 0. - 1) = 1. \\ x_2 &= x_1 + \Delta t f(t_1, x_1) \\ x_2 &= 1. + 0.1 \times (1. + 0.1 - 1) = 1.01 \end{aligned}$$

– Heun's method :

(a) For $x'(t) = x(t)$

$$\begin{aligned}
x_{n+1} &= x_n + \frac{\Delta t}{2}\Big(f(t_n, x_n) + f(t_{n+1}, x_n + \Delta t f(t_n, x_n))\Big) \\
x_0 &= x(0) = 1 \\
x_1 &= x_0 + \frac{\Delta t}{2}\Big(f(t_0, x_0) + f(t_1, x_0 + \Delta t f(t_0, x_0))\Big) \\
x_1 &= x_0 + \frac{\Delta t}{2}(x_0 + x_0 + \Delta t\, x_0) \\
x_1 &= 1. + \frac{0.1}{2}(1. + 1. + 0.1 \times 1.) \\
x_1 &= 1.105 \\
x_2 &= x_1 + \frac{\Delta t}{2}\Big(f(t_1, x_1) + f(t_2, x_1 + \Delta t f(t_1, x_1))\Big) \\
x_2 &= x_1 + \frac{\Delta t}{2}(x_1 + x_1 + \Delta t\, x_1) \\
x_2 &= 1.105 + \frac{0.1}{2}(1.105 + 1.105 + 0.1 \times 1.105) \\
x_2 &= 1.221025
\end{aligned}$$

(b) For $x'(t) = x(t) + t - 1$

$$\begin{aligned}
x_{n+1} &= x_n + \frac{\Delta t}{2}\Big(f(t_n, x_n) + f(t_{n+1}, x_n + \Delta t f(t_n, x_n))\Big) \\
x_0 &= x(0) = 1 \\
x_1 &= x_0 + \frac{\Delta t}{2}\Big(f(t_0, x_0) + f(t_1, x_0 + \Delta t f(t_0, x_0))\Big) \\
x_1 &= x_0 + \frac{\Delta t}{2}(x_0 + t_0 - 1. + x_0 + \Delta t(x_0 + t_0 - 1.) + t_1 - 1.) \\
x_1 &= 1. + \frac{0.1}{2}(1. + 0. - 1. + 1. + 0.1 \times (1. + 0. - 1.) + 0.1 - 1.) \\
x_1 &= 1.005 \\
x_2 &= x_1 + \frac{\Delta t}{2}\Big(f(t_1, x_1) + f(t_2, x_1 + \Delta t f(t_1, x_1))\Big) \\
x_2 &= x_1 + \frac{\Delta t}{2}(x_1 + t_1 - 1. + x_1 + \Delta t(x_1 + t_1 - 1.) + t_2 - 1.) \\
x_2 &= 1.005 + \frac{0.1}{2}(1.005 + 0.1 - 1. + 1.005 + 0.1 \times (1.005 + 0.1 - 1.) + 0.2 - 1.) \\
x_2 &= 1.005 + \frac{0.03205}{2} = 1.021025
\end{aligned}$$

– second-order Runge-Kutta method :

(a) For $x'(t) = x(t)$

$$\begin{aligned}
x_{n+1} &= x_n + \Delta t\, f(t_n + \frac{\Delta t}{2}, x_n + \frac{\Delta t}{2} f(t_n, x_n)) \\
x_0 &= x(0) = 1 \\
x_1 &= x_0 + \Delta t\, f(t_0 + \frac{\Delta t}{2}, x_0 + \frac{\Delta t}{2} f(t_0, x_0)) \\
x_1 &= x_0 + \Delta t\, (x_0 + \frac{\Delta t}{2} x_0) \\
x_1 &= 1. + 0.1 \times (1. + \frac{0.1}{2} \times 1.) = 1.105 \\
x_2 &= x_1 + \Delta t\, f(t_1 + \frac{\Delta t}{2}, x_1 + \frac{\Delta t}{2} f(t_1, x_1)) \\
x_2 &= x_1 + \Delta t\, (x_1 + \frac{\Delta t}{2} x_1) \\
x_2 &= 1.105 + 0.1 \times (1.105 + \frac{0.1}{2} \times 1.105) \\
x_2 &= 1.221025
\end{aligned}$$

(b) For $x'(t) = x(t) + t - 1$

$$\begin{aligned}
x_{n+1} &= x_n + \Delta t\, f(t_n + \frac{\Delta t}{2}, x_n + \frac{\Delta t}{2} f(t_n, x_n)) \\
x_0 &= x(0) = 1 \\
x_1 &= x_0 + \Delta t\, f(t_0 + \frac{\Delta t}{2}, x_0 + \frac{\Delta t}{2} f(t_0, x_0)) \\
x_1 &= x_0 + \Delta t\, (x_0 + \frac{\Delta t}{2}(x_0 + t_0 - 1.) + t_0 + \frac{\Delta t}{2} - 1.) \\
x_1 &= 1. + 0.1 \times (1. + \frac{0.1}{2}(1. + 0. - 1.) + 0. + \frac{0.1}{2} - 1.) = 1.005 \\
x_2 &= x_1 + \Delta t\, f(t_1 + \frac{\Delta t}{2}, x_1 + \frac{\Delta t}{2} f(t_1, x_1)) \\
x_2 &= x_1 + \Delta t\, (x_1 + \frac{\Delta t}{2}(x_1 + t_1 - 1.) + t_1 + \frac{\Delta t}{2} - 1.) \\
x_2 &= x_1 + \Delta t\, ((x_1 + t_1 - 1.)(1. + \frac{\Delta t}{2}) + \frac{\Delta t}{2}) \\
x_2 &= 1.021025
\end{aligned}$$

2. Compare the obtained values with the exact solution : which method seems the more accurate, the less accurate ?

The exact solution for the first differential equation at time $T = 0.2$ is $e^{(0.2)} \approx 1.2214027$. The exact solution for the first differential equation at time $T = 0.2$ is $e^{(0.2)} + \frac{(0.2)^2}{2} - 0.2 \approx 1.0414027$. The less accurate method is the Euler method, the more accurate method are the Heun's method and the second order Runge-Kutta method. In passing you can notice that for these differential equations the Heun's method and the second order Runge-Kutta method yield exactly the same results.