Exercises about first order differential equations

1 Exercice 1

We consider the differential equation :

$$x^2y'(x) + 2x\,y(x) = \sin(2x)$$

Recognize the derivative of a product in order to find the general solution of this equation.

We notice that

$$x^2y' + 2x\,y = (x^2y(x))'$$

Therefore,

$$\begin{aligned}(x^2y(x))' &= \sin(2x)\\ x^2y(x) &= -\frac{\cos(2x)}{2} + c\\ y(x) &= -\frac{\cos(2x)}{2x^2} + \frac{c}{x^2}\end{aligned}$$

2 Exercice 2

We consider a small bay connected to the ocean by means of a narrow channel. The aim of the exercise is to explore how the water level in the bay evolves when the ocean rises and fall due to tides.

We assume that, over a small time interval, the water level in the bay increases proportionally to the difference between the ocean level and the bay level, and to the length of the small time interval.

Let us denote $y(t)$ the heigh of the ocean, and $x(t)$ the water level in the bay.

1. *What is the first order differential equation satisfied by x ?*

If we denote dx the small variation of water level in the bay during the small time interval dt, we can write

$$dx = k\,(y - x)dt$$

which leads to the differential equation

$$x'(t) = k\,(y(t) - x(t)).$$

2. *We assume that the tides happen every 4π hours. If the ocean height is given by $y(t) = \cos(\omega t)$, what value does ω take ?*

The function $y(t)$ should be 4π-periodic. As the cos function is 2π-periodic, ωt should vary between 0 and 2π when t varies between 0 and 4π. Therefore $\omega = \frac{1}{2}$.

3. *Check that the following formula is true :*

$$\int e^{kt}\cos(\omega t)\,dt = \frac{1}{k^2+\omega^2}e^{kt}(k\cos(\omega t)+\omega sin(\omega t))+c$$

We start from the right-hand side of the equation and prove that its derivative is equal to $e^{kt}\cos(\omega t)$, which means that the above equation is true.

$$\begin{aligned}(\frac{1}{k^2+\omega^2}e^{kt}(k\cos(\omega t)+\omega\sin(\omega t))+c)' &= \frac{1}{k^2+\omega^2}k\,e^{kt}(k\cos(\omega t)+\omega\sin(\omega t))\\ &+\frac{1}{k^2+\omega^2}e^{kt}(-k\,\omega\sin(\omega t)+\omega^2\cos(\omega t))\\ &=\frac{1}{k^2+\omega^2}e^{kt}(k^2+\omega^2)\cos(\omega t)=e^{kt}\cos(\omega t)\end{aligned}$$

where c is a constant representing the boundary terms of the integration by parts.

4. *Solve the differential equation using integrating factors.*

$$x'(t)+kx(t)=k\,cos(\omega t)$$

We look for u such that $ux'+kux=(ux)'$: such a function satisfies also : $kux=u'x$. Consequently it is given by the formula

$$u=Ce^{\int kdt}=Ce^{k\,t}$$

We multiply the differential equation by u

$$\begin{aligned}Ce^{k\,t}x'(t)+kCe^{k\,t}x(t)&=k\,cos(\omega t)Ce^{k\,t}\\ (Ce^{k\,t}x)'&=k\,C\cos(\omega t)e^{k\,t}\\ Ce^{k\,t}x&=\int k\,C\cos(\omega s)e^{k\,s}ds\\ x&=e^{-k\,t}\int k\,\cos(\omega s)e^{k\,s}ds\\ x&=e^{-k\,t}\int k\,\cos(\frac{s}{2})e^{k\,s}ds\end{aligned}$$

With the formula of the previous question we can thus write

$$\begin{aligned}x&=e^{-k\,t}k\Big(\frac{1}{k^2+\frac{1}{4}}e^{kt}(k\cos(\frac{t}{2})+\frac{1}{2}\sin(\frac{t}{2}))+c\Big)\\ x&=\frac{4}{4k^2+1}\Big(k^2\cos(\frac{t}{2})+\frac{1}{2}k\sin(\frac{t}{2})\Big)+ce^{-k\,t}k\\ x&=\frac{\Big(4k^2\cos(\frac{t}{2})+2k\sin(\frac{t}{2})\Big)}{4k^2+1}+ce^{-k\,t}k\end{aligned}$$

We can assume that $c=0$ because otherwise the water level would tend to infinity in a large future or past, which is unrealistic.

5. *Your solution should have the form $a\cos(\omega t) + b\sin(\omega t)$ for some constants a and b. Check it by inserting this expression in the differential equation and solving a and b.*
Let us denote $y(t) = a\cos(\omega t) + b\sin(\omega t)$.

$$\begin{aligned} y'(t) &= -a\,\omega\sin(\omega t) + b\,\omega\cos(\omega t) \\ y'(t) + k\,y(t) &= -a\,\omega\sin(\omega t) + b\,\omega\cos(\omega t) + a\,k\cos(\omega t) + b\,k\sin(\omega t) \\ y'(t) + k\,y(t) &= (a\,k + b\,\omega)\cos(\omega t) + (b\,k - a\,\omega)\sin(\omega t) \end{aligned}$$

We need a and b to satisfy

$$\begin{aligned} b\,k - a\,\omega = 0 &\text{ and } a\,k + b\,\omega = k \\ b = a\frac{\omega}{k} &\text{ and } a\,k + a\frac{\omega^2}{k} = k \\ b = a\frac{\omega}{k} &\text{ and } a\frac{\omega^2 + k^2}{k} = k \\ a = \frac{k^2}{\omega^2 + k^2} &\text{ and } b = \frac{\omega}{k}\frac{k^2}{\omega^2 + k^2} \\ a = \frac{4k^2}{1 + 4k^2} &\text{ and } b = \frac{1}{2k}\frac{4k^2}{1 + 4k^2} = \frac{2k}{1 + 4k^2} \end{aligned}$$

Thus

$$y(t) = \frac{4k^2}{1 + 4k^2}\cos(\frac{t}{2}) + \frac{2k}{1 + 4k^2}\sin(\frac{t}{2})$$

3 Exercice 3

1. *Bernouilli equations are differential equations of the form*

$$y' + p(x)y = q(x)y^n, \text{ with } n \neq 1.$$

Show that this kind of differential equation becomes linear if one makes the change of variables $u = y^{1-n}$. (Hint : divide both sides by y^n).

$$\begin{aligned} y' + p(x)y &= q(x)y^n \\ \frac{y'}{y^n} + p(x)\frac{y}{y^n} &= q(x) \\ -\frac{1}{n-1}(\frac{1}{y^{n-1}})' + p(x)\frac{y}{y^n} &= q(x) \\ -\frac{1}{n-1}u' + p(x)u &= q(x) \end{aligned}$$

2. *Solve the following Bernouilli equations :*

$$\begin{aligned} y' + y &= 2xy^2 \\ x^2y' - y^3 &= xy \end{aligned}$$

The first equation becomes

$$u' - u = -2x$$

with $u = y^{1-2} = \dfrac{1}{y}$. We solve it the usual way :

$$\begin{aligned}
u' - u &= -2x \\
Ce^{-x}\, u' - Ce^{-x}\, u &= -2xCe^{-x} \\
(Ce^{-x}\, u)' &= -2xCe^{-x} \\
e^{-x}\, u &= \int -2se^{-s} ds + c \\
e^{-x}\, u &= 2xe^{-x} + 2e^{-x} + c \text{ (because of an integration by parts)} \\
u &= 2x + 2 + ce^{x} \\
y(x) &= \frac{1}{2x + 2 + ce^{x}}
\end{aligned}$$

The second equation reads also

$$y' - \frac{y}{x} = \frac{y^3}{x^2}$$

It becomes :

$$-\frac{u'}{2} - \frac{u}{x} = \frac{1}{x^2}$$

with $u = y^{1-3} = \dfrac{1}{y^2}$.

We re-write it under the form

$$u' + 2\frac{u}{x} = -\frac{2}{x^2}$$

and solve it the classical way : we look for a function p such that

$$\begin{aligned}
u'p + 2\frac{u}{x}p &= (pu)' \\
p' &= 2\frac{p}{x}
\end{aligned}$$

We find $p(x) = Cx^2$.

$$\begin{aligned}
Cx^2u' + 2\frac{u}{x}Cx^2 &= -\frac{2}{x^2}Cx^2 \\
x^2u' + 2ux &= -2 \\
(x^2u)' &= -2 \\
x^2u &= -2x + c \\
u(x) &= \frac{-2x + c}{x^2} \\
y(x) &= \pm\frac{x}{\sqrt{-2x + c}}
\end{aligned}$$