Exercises about first order differential equations

1 Exercice 1

We consider the differential equation :

$$x^2 y'(x) + 2x\, y(x) = \sin(2x)$$

Recognize the derivative of a product in order to find the general solution of this equation

2 Exercice 2

We consider a small bay connected to the ocean by means of a narrow channel. The aim of the exercise is to explore how the water level in the bay evolves when the ocean rises and fall due to tides.

We assume that, over a small time interval, the water level in the bay increases proportionally to the difference between the ocean level and the bay level, and to the length of the small time interval.

Let us denote $y(t)$ the heigh of the ocean, and $x(t)$ the water level in the bay.

1. What is the first order differential equation satisfied by x ?
2. We assume that the tides happen every $4\,\pi$ hours. If the ocean height is given by $y(t) = \cos(\omega t)$, what value does ω take ?
3. Check that the following formula is true :

$$\int e^{kt} \cos(\omega t)\, dt = \frac{1}{k^2 + \omega^2} e^{kt} (k \cos(\omega t) + \omega sin(\omega t)) + c$$

4. Solve the differential equation using integrating factors.
5. Your solution should have the form $a \cos(\omega t) + b \sin(\omega t)$ for some constants a and b. Check it by inserting this expression in the differential equation and solving a and b.

3 Exercice 3

1. Bernouilli equations are differential equations of the form

$$y' + p(x)y = q(x)y^n, \text{ with } n \neq 1.$$

Show that this kind of differential equation becomes linear if one makes the change of variables $u = y^{1-n}$. (Hint : divide both sides by y^n).

2. Solve the following Bernouilli equations :

$$y' + y = 2xy^2$$
$$x^2y' - y^3 = xy$$