Topic 1:
Method to synthesize analog filters

I- Method to synthesize a “Butterworth” active low-pass filter

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II-2. Practical simulation

III- Applied exercise *(taken from an S3 exam paper)*

Aims:

The aim of this TA is to:

- be able to **synthesize an analog filter** from a given template,
- be able to calculate the elements of first- and second-order filters that satisfy the **characteristic angular frequency and quality factor** values imposed by specifications,
- understand the advantages of active analog filters based on OP-AMPs but also **their limitations**.

Prerequisites:

S1 and S2 courses on Circuits and Electronics. Bode diagram. Transfer function. Complex numbers. Logarithms.

I- Method to synthesize a “Butterworth” active low-pass filter

I-1. Theoretical study

I-1-a. Calculating order and characteristic frequency

Our aim is to synthesize a “Butterworth” analog low-pass filter type using the template shown in Figure 1. Other types of filter (Bessel, Chebyshev, Cauer, etc.) are too difficult to synthesize analytically and require the use of a computer. In order to study their properties and compare their performances, these other types of filter, especially Bessel and Chebyshev, will be dealt with in the PA.

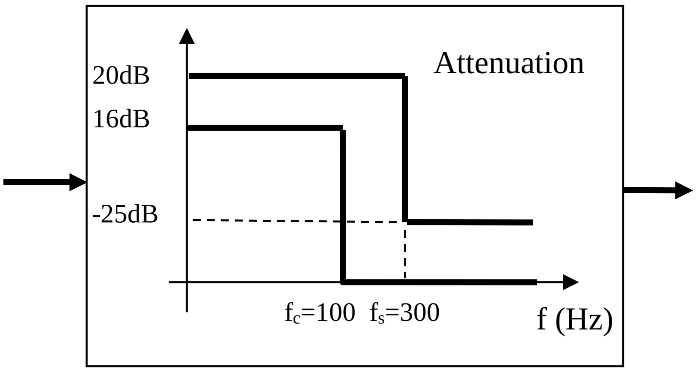

Figure 1: *Template for low-pass filter and digital values in specifications*

Frequencies f_c and f_s refer to the end of the **pass-band** and the start of the **stop-band** respectively.

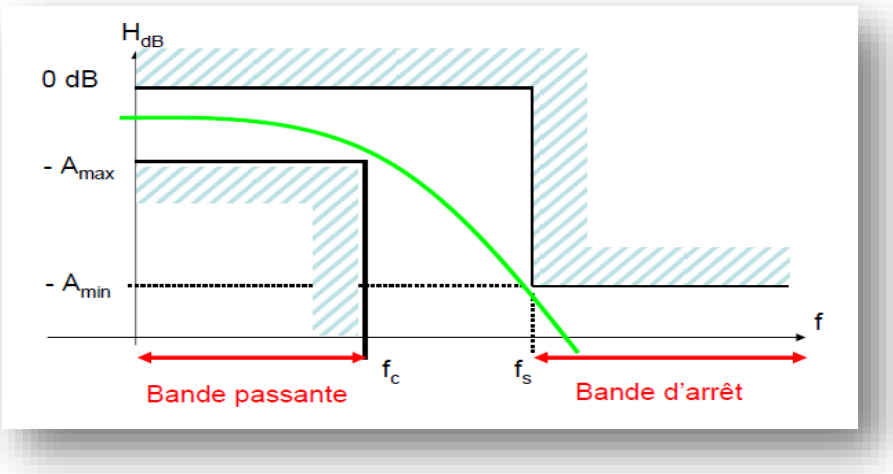

Figure 2: *Quantities associated with the normalized filter template*

NB: Quantities A_{max} and A_{min} correspond to maximum acceptable pass-band ripple and minimal acceptable stop-band attenuation respectively.

Question 1: The filter is produced by cascading together a low-pass filter and an amplifier with a gain of 10. Complete the digital values in the template of the low-pass filter in Figure 3.

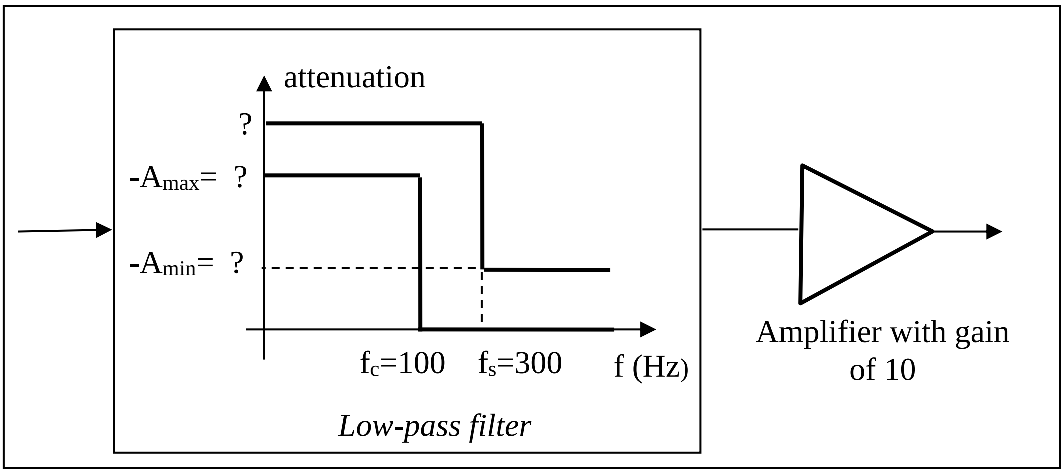

Figure 3: *Filter produced by cascading together a low-pass filter and an amplifier with a gain of 10.*

Question 2: The gain module of a low-pass Butterworth filter of order n and cutoff angular frequency ω_0 at -3dB is written:

$$|H(j\omega)| = \frac{1}{\left(1+\left[\frac{\omega}{\omega_0}\right]^{2n}\right)^{1/2}} \qquad (1)$$

- What values is $|H(j\omega)|$ tending towards when $\omega \to 0$ and $\omega \to \infty$ and what is the value of $|H(j\omega)|$ for the specific angular frequency $\omega = \omega_0$ also called the characteristic angular frequency?
- Explain why ω_0 corresponds to the cutoff angular frequency value at -3dB **(attention: this is not the case for all filters!).**

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Question 3: From the completed template in Figure 3 and equation (1), we obtain **2 equations with 2 unknowns** for the two specific points f_C and f_S. Derive the expression for **order n** and **angular frequency** ω_0. Start by calculating n, then select the integer just above this value and extract ω_0.

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I-1-b. Synthesizing the filter

Question 4: If your calculations are correct, you should find that the low-pass filter in Figure 2 represents a cascading together of two second-order filters and one first-order filter. Using the table in Appendix 1:

- for each of the second-order filters, give the value of the overvoltage coefficient Q and the characteristic angular frequency ω_0,
- for the first-order filter, give the value of the characteristic angular frequency ω_0.

	value of Q	**value of ω_0**
first second-order filter		
second second-order filter		
first-order filter		

<u>NB</u>: Remember that the transfer functions $H(p)$ (Laplace transform) for the first-order and second-order filters respectively are written in the following forms:

$$H(p) = \frac{\omega_0}{p + \omega_0} \qquad (2)$$

$$H(p) = \frac{\omega_0^2}{p^2 + p\frac{\omega_0}{Q} + \omega_0^2} \qquad (3)$$

Question 5: The first-order filter is made according to the diagram in Figure 4. The OP-AMP is assumed to be ideal.

- Check that the filter transfer function $H(p) = \frac{V_S(p)}{V_e(p)}$ is similar to that in equation 2 above.
- What is the advantage of putting in an OP-AMP since the gain is the same without an OP-AMP?

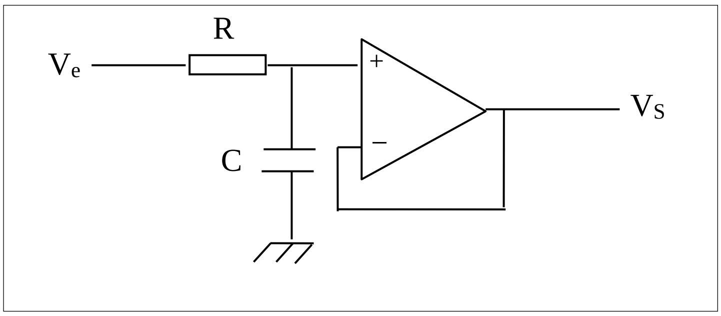

Figure 4: *First-order low-pass filter*

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Question 6: Assume $R = 10k\Omega$, calculate the value of C.

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The filter in Figure 1 requires a gain-of-10 amplifier, this gain can be included in the previous filter, as shown in the diagram in Figure 5.

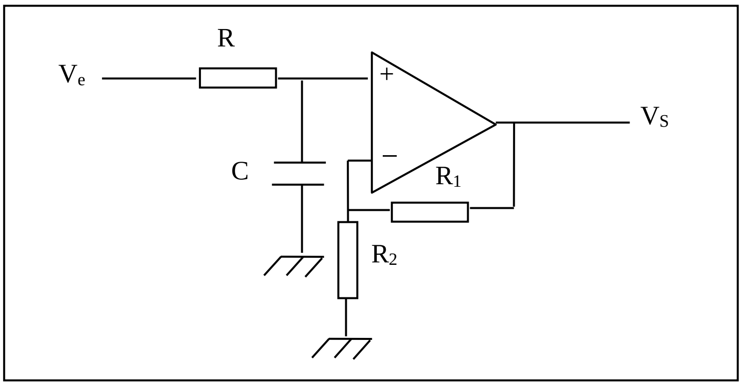

Figure 5: *Low-pass filter with gain for* $f \to 0$

Question 7:

- Show that the transfer function of this filter is written in the form:

$$H(p) = A_{LP} \frac{\omega_0}{p + \omega_0} \qquad (4)$$

where A_{LP} is the gain at zero frequency.

- Give the expression of A_{LP} as a function of R_1 and R_2. Choose $R_2 = 1k\Omega$. Give the value R_1 in order to obtain $A_{LP} = 10$.

The two second-order filters are made from active cells known as "Sallen-Key" (see diagram in Figure 6). The filter transfer function $H(p) = \frac{V_S(p)}{V_e(p)}$ shown in Figure 6 is given by equation 3. To obtain this, write the equations at nodes A and B.

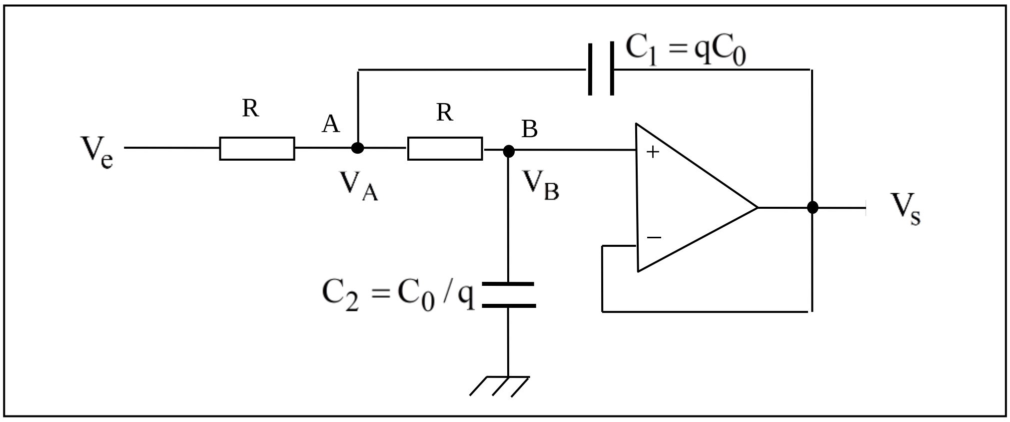

Figure 6: *Sallen-Key active cell to make a second-order filter*

Question 8: **As the OP-AMP is assumed ideal**, what approximately is the potential V_B at point B in relation to ground potential? Then write the equations at nodes A and B as a function of V_e, V_s, V_A, R and the functional admittances qC_0p and $\frac{C_0}{q}p$.

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Question 9: To obtain $H(p) = \frac{V_s(p)}{V_e(p)}$, extract the potential V_A from one of the 2 equations then replace it in the other equation. This should give an equation identical to equation 3. Identify the expressions of Q as a function of q then of ω_0 as a function of R and C_0.

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Question 10: $R = 10k\Omega$ is given. Calculate capacitances C_1 and C_2 to place on the circuit for each filter.

	value of Q	**value of ω_0**	**R**	C_1	C_2
first second-order filter			$10k\Omega$		
second second-order filter			$10k\Omega$		

Question 11:

- Give the full diagram for the filter.
- Can the position of the filter blocks be reversed in the final circuit? Justify your answer.

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Question 12:

- Write the expression for the complex gains of each filter for frequencies very much higher than the cutoff frequency f_0.
- Then write the full gain of the filter in dB, i.e. $20\log_{10}|H(j\omega)|$. From this determine the attenuation slope (in dB/decade) for frequencies $f >> f_0$.

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Question 13: What are the phase differences between output and input signals for $f \to 0$ and $f \to \infty$? Justify your answer.

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I-2. Practical simulation

I-2-a. Presentation of Tina-TI software

Tina-TI (available as a free download on the Texas Instruments website www.ti.com) is an electronic circuit simulation software package. It includes a SPICE model library of OP-AMPs, instrumentation amplifiers, comparators, voltage regulators, etc., produced by Texas Instruments. After entering the circuit, different analyses are possible (Figures 7 and 8):

- DC analysis (analysis of magnitudes in continuous regime)
- AC analysis (analysis of magnitudes in variable regime)
- Transient analysis (analysis of magnitudes in transient regime)
- Fourier analysis (frequency domain analysis)
- Noise analysis

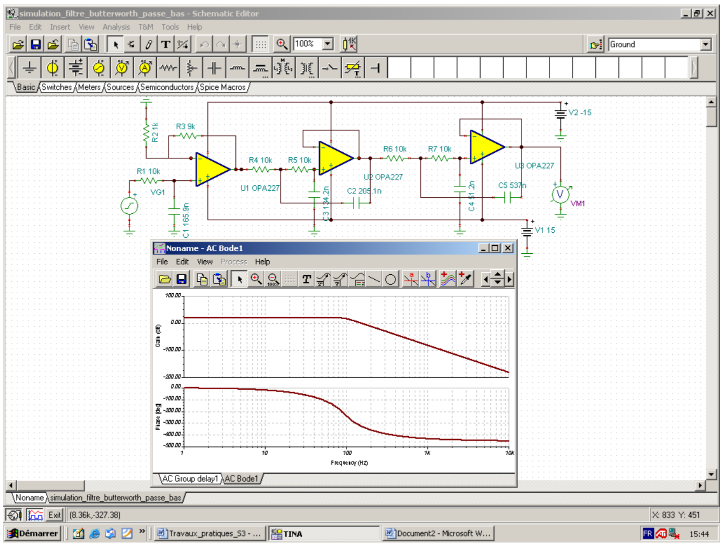


Figure 7: *Tina-TI software sample screen: circuit schematic and result of AC analysis simulation (Bode diagram)*

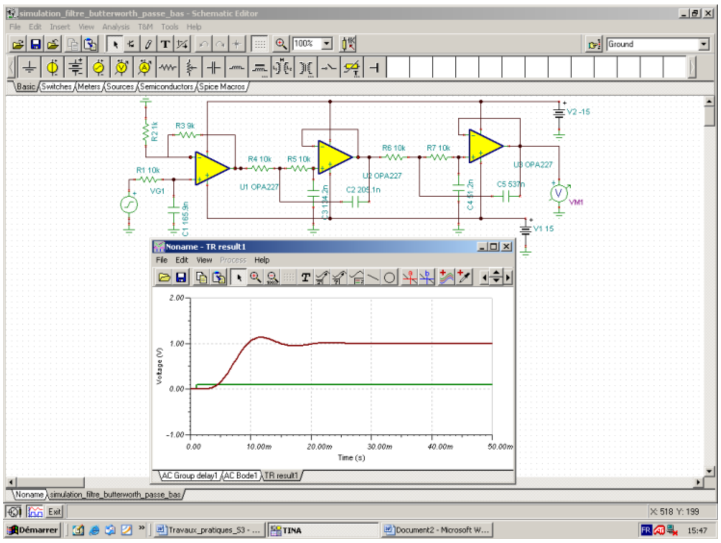


Figure 8: *Tina-TI software sample screen: circuit schematic and result of Transient Analysis simulation*

The software is fairly easy to use and user-friendly.

- To select a component, scroll down and click OK in the component selection window then drag it into position in the circuit workspace.
- To set the value of a component, double-click on it and fill in the required fields.
- Draw in wires between the components and move on to one of the analyses mentioned above.

NB: Tina-TI software only simulates active circuits. However, this is not a problem as you can simply add a follower type assembly, for example, in order to analyze a circuit with only passive components.

I-2-b. Studying the filter in harmonic regime

➢ **Enter** the circuit schematic of the Butterworth filter, adding the values calculated previously. Put the first-order filter in place first.

Procedure for schematic entry:

- Select OP-AMPs: Horizontal navigation bar → Spice Macros → Operational Amplifiers → OPA227
- Select passive components (Resistor, Capacitor, Ground, Battery): Horizontal navigation bar → Basic
- Select generator: Horizontal navigation bar → Sources → Voltage Generator
- Select measurement equipment: Horizontal navigation bar → Meters → Volt Meter

➢ **Study in harmonic regime**:

Procedure to obtain complex gain module and phase:

- Horizontal navigation bar → Analysis → AC Analysis → AC Transfer Characteristic
- Complete the fields in the window (Start Frequency, End Frequency, Number of points, Sweep type Logarithmic, Diagram Amplitude & Phase)

I-2-c. Applying the results

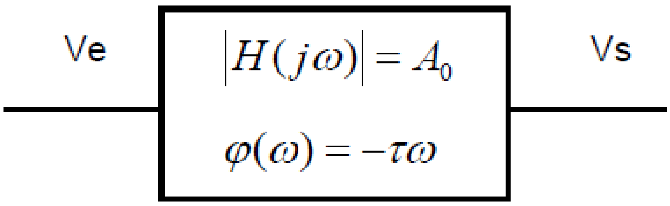
Question 14: Using the zoom function and cursors, determine:

- gain at low frequencies ($f \to 0$)
- cutoff frequency f_{-3dB}
- attenuation slope in dB/decade for $f \to \infty$
- phase shift for $f \to \infty$
- cutoff frequency f^1_{-3dB} of the first-order filter placed at the top of the filter, give the value of k in $f^1_{-3dB} = kf_{-3dB}$. Compare this to the theoretical value.
- Compare the values obtained from the simulation with those calculated previously.

I-2-d. Determining group delay

The purpose of a low-pass filter is to eliminate frequencies in the stop band and to allow frequencies in the pass-band to pass through. If the signal applied to a filter input includes several frequencies in its pass-band, it is clearly better for these frequencies to have the **same delay** so that the useful signal is simply delayed and is therefore not distorted (Figures 9 and 10).

Exemple : $v_e(t) = A1\cos(\omega_1 t)+A2\cos(\omega_2 t)+A3\cos(\omega_3 t)$ avec $\omega_0 >> \omega_{1,2,3}$



$|H(j\omega)| = A_0$

$\varphi(\omega) = -\tau\omega$

$v_s(t) = A_0\,[A1\cos(\omega_1 t+\phi_1))+A2\cos(\omega_2 t+\phi_2)+A3\cos(\omega_3 t+\phi_3)]$

$v_s(t) = A_0\,[A1\cos(\omega_1 t-\tau\omega_1))+A2\cos(\omega_2 t-\tau\omega_2)+A3\cos(\omega_3 t-\tau\omega_3)]$

$v_s(t) = A_0\,[A1\cos(\omega_1\,(t-\tau)\,)+A2\cos(\omega_2\,(t-\tau)\,)+A3\cos(\omega_3\,(t-\tau)\,)]$

$v_s(t) = A_0\, v_e(t-\tau)$

Figure 9: *Notion of group delay on a multi-frequency input signal*

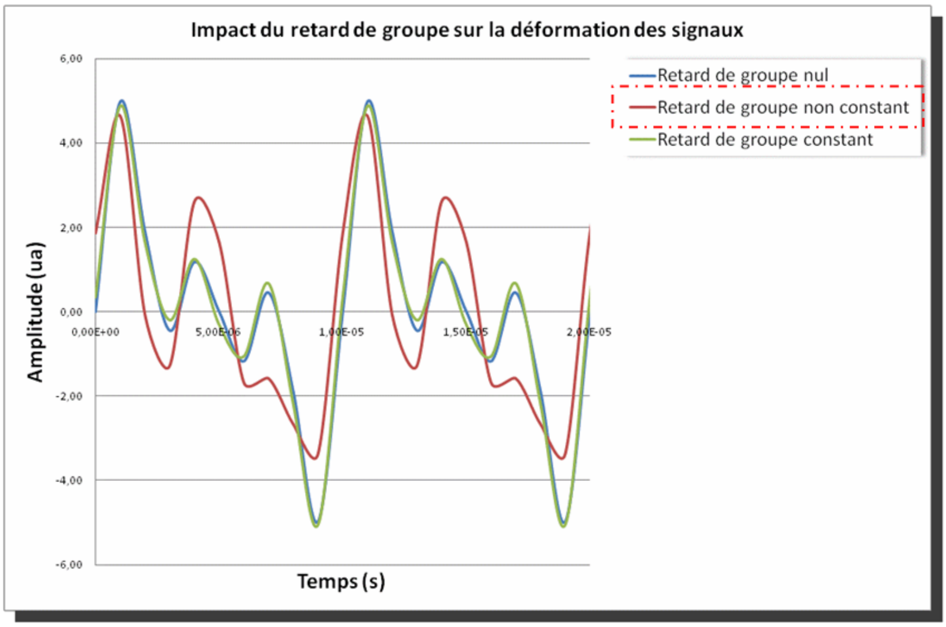
- All frequencies pass through the filter in a time that is exactly equal.
- As you will see in the PA, **only the Bessel filter** has this capability, minimizing the distortion that a complex signal is subject to during a filter operation.

For a given angular frequency $\omega = \frac{2\pi}{T}$, the delay is $\frac{T}{2\pi}\varphi(\omega) = \frac{\varphi(\omega)}{\omega}$ where T is the time period and $\varphi(\omega)$ is the filter phase shift at angular frequency ω.

To obtain a constant delay, whatever the value of ω, then it must be the case that $\frac{\varphi(\omega)}{\omega} = C^{te}$, or phase $\varphi(\omega)$ must be proportional to the angular frequency, i.e. $\varphi(\omega) = C^{te}\omega$, then the delay would be equal to C^{te} for all angular frequencies. No analog filter is able to ensure a constant delay for all angular frequencies, but some filters have a more linear phase than others, and this is the case for **Bessel filters**, which we will study next.

$x(t) = 2.\sin 2\pi f_1(t) + 2.\sin 2\pi f_2(t) + 2.\sin 2\pi f_3(t)$
$f_1 = 100\,kHz,\ f_2 = 200\,kHz,\ f_3 = 300\,kHz$

$y(t) = 2.\sin 2\pi f_1(t-\tau_1) + 2.\sin 2\pi f_2(t-\tau_2) + 2.\sin 2\pi f_3(t-\tau_3)$
$f_1 = 100\,kHz,\ f_2 = 200\,kHz,\ f_3 = 300\,kHz$



Impact du retard de groupe sur la déformation des signaux=> Impact of group delay on signal distortion Retard de groupe nul=> Zero group delay Retard de groupe non constant=> Non-constant group delay Retard de groupe constant=> Constant group delay Amplitude=>Amplitude Temps=>Time

Figure 10: *Illustration of the influence of group delay on a multi-frequency input signal*

To define the linearity of the phase as a function of frequency, we introduce group delay $\tau(\omega) = -\frac{d\varphi}{d\omega}$, derived from the phase in relation to angular frequency. If the phase is proportional to the angular frequency, then τ is a constant (see Figure 11).

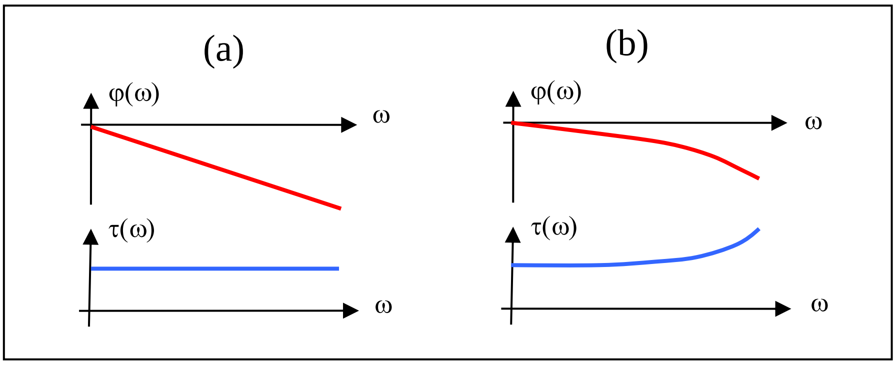


Figure 11: *(a) linear phase → constant group delay → non-distorted delayed signal, (b) non-linear phase → non-constant group delay → distorted signal*

To obtain the group delay, proceed as follows:

- Horizontal navigation bar → Analysis → AC Analysis → AC Transfer Characteristic: complete the fields in the window (Start Frequency, End Frequency, Number of points, Sweep type Logarithmic, Group Delay)

Question 15:

- Give the group delay for $f \rightarrow 0$
- Is the group delay constant in the pass-band?
- What is the maximum value of group delay in the pass-band?

I-2-e. Step response or step function response

Procedure to follow to obtain the step response:

- Double-click on generator →
  Signal Unit = Step, Amplitude = 100mV, Start of edge = 0s
- Horizontal navigation bar → Analysis → AC Analysis → Transient: complete the fields in the window (Start display, End display, Draw excitation)

Question 16:

- Give the settling time at 1%.
- What is the delay at 50% of the final value? You must check that the delay is of the same order of magnitude as the group delay measured previously.

Definition of "settling time" (Figure 12):
*The **settling time** of an amplifier or other output device is the time lapse from the application of an ideal instantaneous step input to the time at which the amplifier output has entered and remained within a specified error band, usually symmetrical around the final value.*

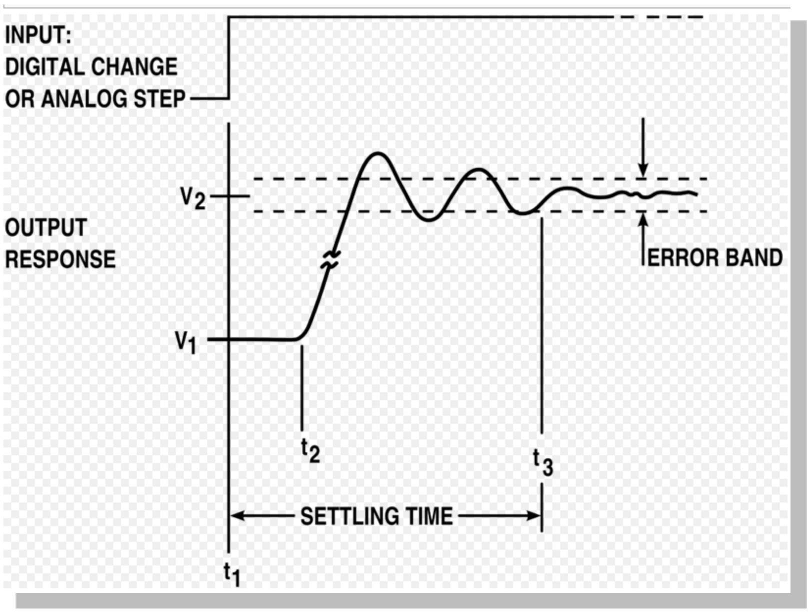


Figure 12: *Definition of "settling time"*

NB: Step response and frequency response are obviously linked (TF (derived from the step response) = frequency response). We observe that the step response of the Butterworth filter is "fairly" disturbed, which is directly linked with the fact that group delay varies in the pass-band. We shall see that the step response of the Bessel filter, on the other hand, is much less disturbed.

II- Effect of limitations of using OP-AMPs

II-1. Theoretical study – non-ideal OP-AMP

In the previous questions, **OP-AMPs were assumed to be ideal with an infinite open-loop gain**, i.e. an input-output relation: $V_s = A_d (V^+ - V^-)$ with $A_d \to \infty$.

In practice, the OP-AMP has its own transfer function:

$$V_s = \frac{A_d}{(1 + j\frac{\omega}{\omega_c})}(V^+ - V^-) \qquad (5)$$

where A_d is the static gain (f→0) in difference mode and ω_c is the cutoff angular frequency at -3dB in open loop.

Figure 13 shows the gain $\frac{V_s}{(V^+ - V^-)}$ of OP-AMP TL081 as a function of frequency.

Question 17: Use this figure to determine the values of A_d and f_C.

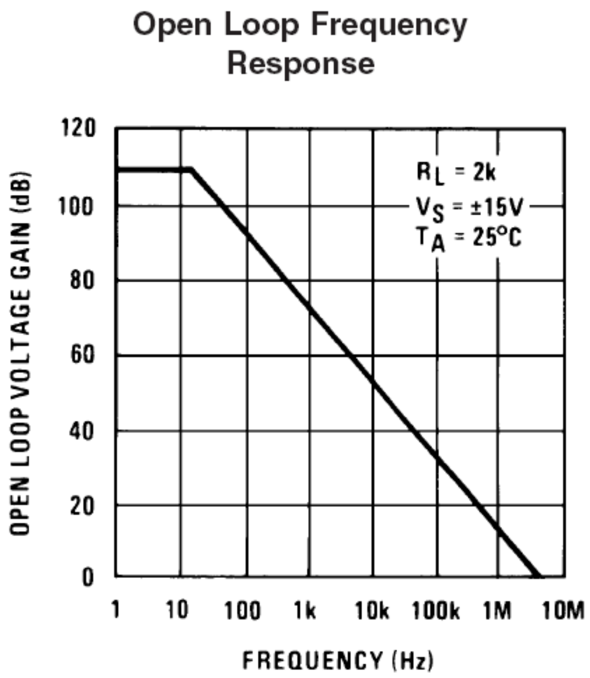


Figure 13: *Open loop gain of OP-AMP TL081*

Question 18: Determine the complex gain $H(j\omega) = \frac{Vs}{Ve}$ of the assembly in Figure 14.

To do this, write the voltage $(V^+ - V^-)$ and transfer it into the expression for OP-AMP open-loop gain.

If you assume $A_d >> (1+\frac{R_1}{R_2})$, then you should obtain an expression that takes the form:

$$H(j\omega) = \frac{Vs}{Ve} \approx \left(1+\frac{R_1}{R_2}\right)\frac{1}{1+j\frac{\omega}{\omega_c'}} \quad \text{with } \omega_c' \approx \omega_c \frac{A_d}{(1+\frac{R_1}{R_2})} \qquad (6)$$

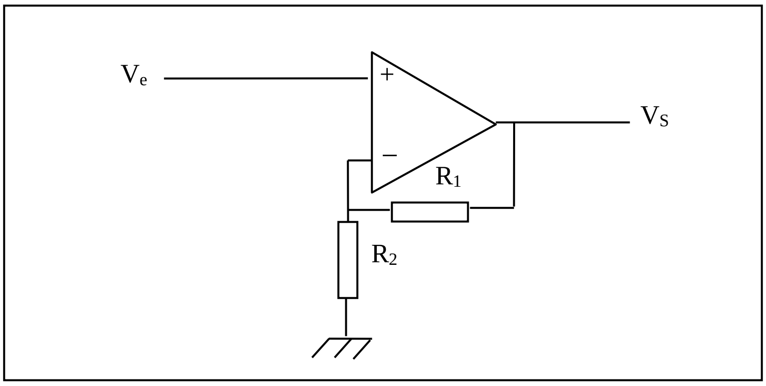


Figure 14: *Operational amplifier with limited pass-band*

Question 19:
What is the role played by this assembly if we take account of its limited pass-band?

Next, we want to make a filter according to the template in Figure 15. The filter is produced by cascading together a filter without static gain followed by an amplifier with gain of 10000, as shown in the general diagram in Figure 3.

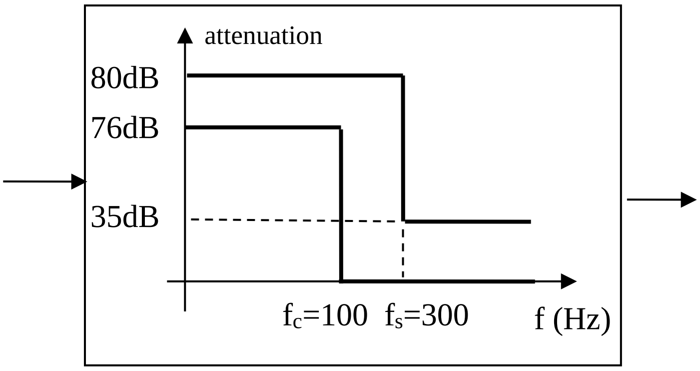


Figure 15: *Template of low-pass filter*

Question 20: The gain of 10000 is achieved with the assembly in Figure 15.

- Calculate the cutoff frequency of the assembly at -3dB.
- Ultimately, what order of filter will be used (4th, 5th or 6th)? Justify your answer. In this specific case, is this an advantage or a disadvantage?

Question 21:
The specifications are modified and require 100dB instead of 80dB, what happens?

To answer this question, we continue to use equation (6) to calculate ω_c' even though the condition $A_d \gg (1+\frac{R_1}{R_2})$ is no longer really satisfied.

II-2. Practical simulation

Modify your general diagram to produce a Butterworth filter that satisfies the template in Figure 16.

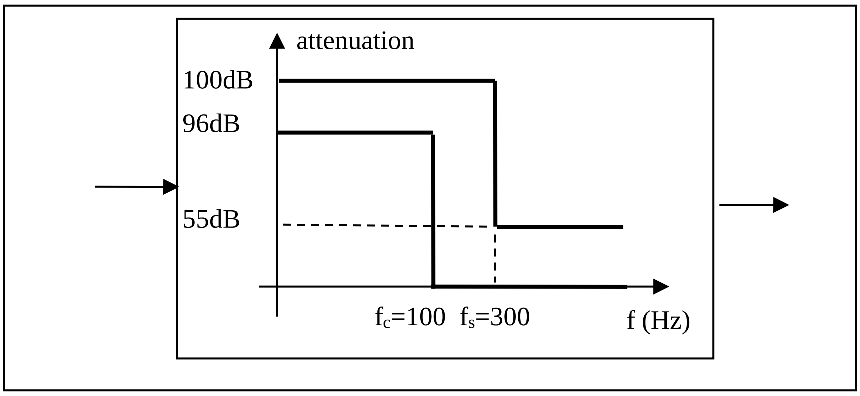


Figure 16: *Template of filter with static gain of 100dB*

Analyze **in AC mode** and determine:

- gain at low frequencies ($f \to 0$)
- cutoff frequency at -3dB
- the attenuation slope in dB/decade
- phase shift for $f \to \infty$
- Determine the cutoff frequency f^{1}_{-3dB} of the first-order filter placed at the top of the filter. Conclusion?

REMEMBER: It is important to remember that an active analog filter is synthesized from specifications which impose a template. This establishes the order (n) and the characteristic angular frequency/frequency of the filter (ω_0). In this TA, only the synthesis of a Butterworth-type filter has been covered as this is the only filter that can be applied using a simple analytical approach. In this case, and only in this case, the cutoff angular frequency is equal to the angular frequency at -3dB.

An active filter of order n consists of active blocks (1st and 2nd order) based on OP-AMPs. These blocks correspond to different types of "circuit" architecture (low-pass, high-pass, bandpass, etc.). Usually we use "Sallen-Key" cells, where determining the values of active components (R, C) must be based on knowing the characteristic angular frequency (ω_0) and the quality factor (Q) of each block.

Filter behavior can be analyzed from a point of view of frequency (harmonic regime) and time (transient or indicial state). We have also seen that group delay is an important notion as it determines the degree of distortion of a multi-frequency signal. In this context, the most interesting filter is the Bessel filter (known as linear phase) as it has a group delay which is virtually constant in its pass-band, unlike the Butterworth and Chebyshev filters.

Finally, particular attention should be paid to the use of OP-AMPs, which cannot be assumed to be ideal, especially when the specifications impose a large static gain (>80dB). They then impose their own frequency response (often 1st order) as their gain in differential mode (Ad) cannot always be considered as very high whatever the frequency.

III- Applied exercise *(taken from an S3 exam paper)*

Taking measurements from a filter, asymptotes have been drawn as shown in Figure 1.

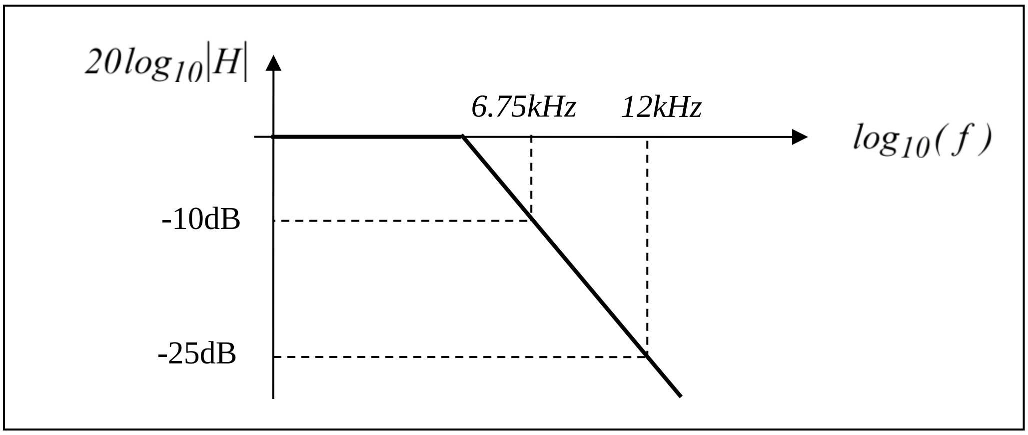


Figure 1: *Plot of the filter gain asymptotes*

Q1- What is the slope in $dB/decade$ when $f \to \infty$? Use this to establish the order of filter.

Q2- We consider the template of a low-pass Butterworth-type filter shown in Figure 2. Determine the order of this filter. The order will be the whole number immediately above the decimal value obtained. Remember the expression for the gain module of a Butterworth filter:

$$|H(j\omega)| = \frac{1}{\left(1+\left[\frac{\omega}{\omega_0}\right]^{2n}\right)^{1/2}}$$

where ω_0 represents the characteristic angular frequency.

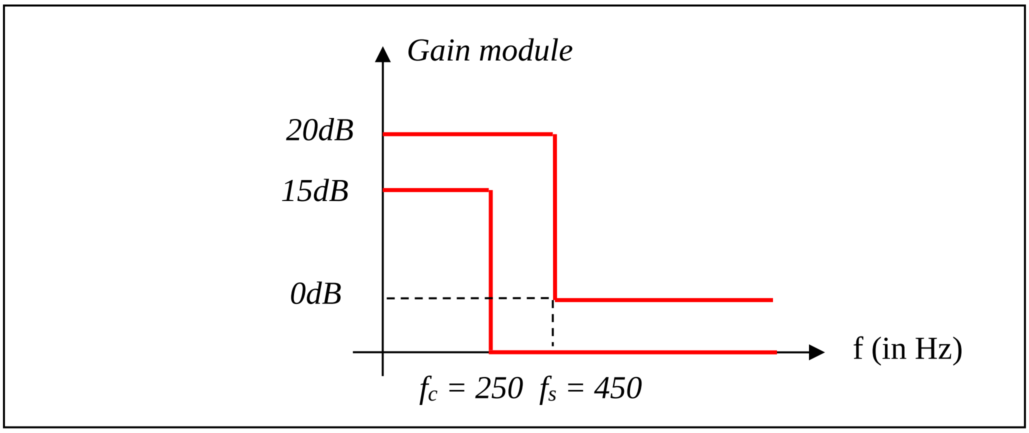


Figure 2: *Template of low-pass filter*

Q3- A low-pass fifth-order Bessel filter has a pass-band at *-3dB* equal to *1kHz* with a group delay of *2ms*. This filter is attacked by the following input signal: $e(t) = 2\cos(2\pi 30t) + 3\cos(2\pi 70t) + 0.5\cos(2\pi 10^4 t)$.

- What is the main characteristic of the group delay for a Bessel filter?
- Give the expression, with justification, of the output signal *s(t)* from the filter.