Spectral Theorem for compact self-adjoint operators

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http://www.u-bordeaux.fr/˜ pjaming/enseignement/M1.html

Master Mathématiques et Applications
M1 Mathématiques fondamentales & M1 Analyse, EDP, probabilités
Lecture : Introduction to spectral analysis

Preliminary warning

This video is a complement to the lecture notes available at

http://www.u-bordeaux.fr/~pjaming/enseignement/M1.html

1. Statement of the Spectral Theorem
2. An example
3. Toolbox

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Aim: give the main tools to prove the spectral theorem

Theorem (Spectral Theorem)

H separable, infinite dimensional, Hilbert space. $T : H \to H$ *compact, self-adjoint operator.*

$\exists\ (e_k)_{k\in\mathbb{N}}$ *orthonormal basis of H;*

$\exists\ (\lambda_k)_{k\in\mathbb{N}}$ *real, $\lambda_k \to 0$;*

$$Tx = \sum_{k\in\mathbb{N}} \lambda_k \langle x, e_k\rangle e_k.$$

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Stability

Throughout H=Hilbert space, $T : H \to H$ bounded linear.

Observation

If $T = T^*$ and $E \subset H$ closed subspace invariant through $T : T(E) \subset E$ then $E^\perp$ is also stable through T

Proof.

$x \in E$, $y \in E^\perp$ then $Tx \in E$ and

$$\langle Ty, x\rangle = \Big\langle \underbrace{y}_{\in E^\perp}, \underbrace{Tx}_{\in E} \Big\rangle = 0$$

that is $Ty \in E^\perp$. □

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Kernel and image

$\ker T$ is stable through T.

Observation

If $T = T^*$, $H = \ker T \oplus \overline{\mathrm{Im}\, T}$, *i.e.* $\ker T = \mathrm{Im}\, T^\perp$ *i.e.* $(\ker T)^\perp = \overline{\mathrm{Im}\, T}$.

Proof.

$x \in \mathrm{Im}\, T^\perp$ means $\forall y \in H$

$$0 = \langle x, Ty \rangle = \langle Tx, y \rangle$$

$\Leftrightarrow Tx = 0$ *i.e.* $x \in \ker T$. □

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Reduction to one-to-one operators

π_0, $\perp$-proj on $\ker T$, $I - \pi_0$ $\perp$-proj on $\overline{\mathrm{Im}\, T}$ then (with previous observation)

$$T = \pi_0 T \pi_0 + (I - \pi_0) T (I - \pi_0)$$

both are self adjoint and compact if T is.
$(I - \pi_0) T (I - \pi_0)$ is $T\ : \overline{\mathrm{Im}\, T} \to \overline{\mathrm{Im}\, T}$ and is one-to-one.

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Spectral theorem - comments

one can also interpret this as T having a matrix of the form

	$\ker T$	E_1		E_2		$\cdots$	
		$e_{1,1},\dots,e_{n_1,1}$		$e_{1,2},\dots,e_{n_2,2}$			
$\ker T$	0	0		0		$\cdots$	
$e_{1,1}$		λ_1	0				
$\vdots$	0	$\ddots$				0	
$e_{n_1,1}$		0	λ_1				
$e_{1,2}$				λ_2	0		
$\vdots$	0			$\ddots$			
$e_{n_2,2}$				0	λ_2		
						$\ddots$	0
	0	0				$\ddots$	
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Eigenvalues are real

Observation

If $T = T^*$, *and* λ *eigenvalue:* $\exists x \in H$, $x \neq 0$ *s.t.* $Tx = \lambda x$ *then* $\lambda \in \mathbb{R}$.

Proof.

$$\begin{aligned} \lambda \|x\|^2 &= \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle \\ &= \langle x, \lambda x\rangle = \bar{\lambda}\|x\|^2. \end{aligned}$$

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Eigenspaces are orthogonal

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If $T = T^*$, $\lambda \neq \mu$, $E_\lambda = \ker(T - \lambda I)$, E_μ *corresponding eigenspaces then* $E_\lambda \perp E_\mu$

Proof.

$x \in E_\lambda, y \in E_\mu$

$$\begin{aligned} \lambda\langle x, y\rangle &= \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle \\ &= \langle x, \mu y\rangle = \bar{\mu}\langle x, y\rangle = \mu\langle x, y\rangle \end{aligned}$$

□

$E_\mu \subset E_\lambda^\perp$ and E_μ=eigenspace for eigenvalue μ of $(I - \pi_\lambda)T(I - \pi_\lambda)$ *i.e.* of T restricted to $E_\lambda^\perp$

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Eigenvalues exist

Lemma

If $T \neq 0$ is compact, self-adjoint the either $\|T\|$ or $-\|T\|$ is an eigenvalue

Proof: $\alpha := \|T\|$ then

$$\begin{aligned}\alpha^2 &= \sup\{\|Tx\|^2 : \|x\| = 1\} = \sup\{\langle Tx, Tx\rangle : \|x\| = 1\}\\ &= \sup\{\left\langle T^2x, x\right\rangle : \|x\| = 1\}.\end{aligned}$$

$\exists x_n$, $\|x_n\| = 1$, $\langle T^2x_n, x_n\rangle \to \alpha^2$.

$$\begin{aligned}\left\|T^2x_n - \alpha^2x_n\right\|^2 &= \left\|T^2x_n\right\|^2 - 2\alpha^2\Re\left\langle T^2x_n, x_n\right\rangle + \alpha^4\|x_n\|^4\\ &\leq \|T\|^4\|x_n\|^2 - 2\alpha^2\Re\left\langle T^2x_n, x_n\right\rangle + \alpha^4\|x_n\|^4\\ &= 2\alpha^4 - 2\alpha^2\Re\left\langle T^2x_n, x_n\right\rangle \to 0.\end{aligned}$$

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Eigenvalues exist

Lemma

If $T \neq 0$ is compact, self-adjoint the either $\|T\|$ or $-\|T\|$ is an eigenvalue

Proof: $\alpha := \|T\|$ then

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Eigenspaces have finite dimension

Observation

If T is compact, each eigenspace E_λ ($\lambda \neq 0$) has finite dimension.

Proof.

$(e_k)_{k\in I}$ O.N.B of E_λ, $Te_k = \lambda e_k$. If $k \neq \ell$, $\|Te_k - Te_\ell\| = |\lambda|\sqrt{2} \neq 0$.
Can not extract convergent subsequence (if I infinite)
Contradicts T compact.
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Thank you for your attention!

Next video: The proof.

http://www.u-bordeaux.fr/~pjaming/enseignement/M1.html

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