Spectral Theorem for compact self-adjoint operators

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http://www.u-bordeaux.fr/˜ pjaming/enseignement/M1.html

Master Mathématiques et Applications
M1 Mathématiques fondamentales & M1 Analyse, EDP, probabilités
Lecture : Introduction to spectral analysis

Preliminary warning

This video is a complement to the lecture notes available at

http://www.u-bordeaux.fr/~pjaming/enseignement/M1.html

1. Statement of the Spectral Theorem
2. An example
3. Toolbox
4. The proof

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Spectral Theorem for compact self-adjoint operators

Aim: give the main tools to prove the spectral theorem

Theorem (Spectral Theorem)

H, infinite dimensional, Hilbert space. $T : H \to H$ *compact, self-adjoint, one-to-one operator.*
$\exists\, (e_k)_{k\in\mathbb{N}}$ *orthonormal basis of H;*
$\exists\, (\lambda_k)_{k\in\mathbb{N}}$ *real, , λ_k decreasing $\lambda_k \to 0$;*

$$Tx = \sum_{k\in\mathbb{N}} \lambda_k \langle x, e_k\rangle e_k.$$

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Spectral theorem - comments

T has a matrix of the form

$$
\begin{array}{c|c||cc|cc|ccc}
 & \ker T & \multicolumn{2}{c}{E_1} & \multicolumn{2}{c}{E_2} & & \cdots & \\
 & & \multicolumn{2}{c}{e_{1,1},\dots,e_{n_1,1}} & \multicolumn{2}{c}{e_{1,2},\dots,e_{n_2,2}} & & & \\
\hline
\ker T & 0 & \multicolumn{2}{c}{0} & \multicolumn{2}{c}{0} & & \cdots & \\
\hline\hline
e_{1,1} & & \lambda_1 & 0 & & & & & \\
\vdots & 0 & \multicolumn{2}{c}{\ddots} & & & & 0 & \\
e_{n_1,1} & & 0 & \lambda_1 & & & & & \\
e_{1,2} & & & & \lambda_2 & 0 & & & \\
\vdots & 0 & & & \multicolumn{2}{c}{\ddots} & & & \\
e_{n_2,2} & & & & 0 & \lambda_2 & & & \\
 & & & & & & \ddots & & 0 \\
 & 0 & \multicolumn{2}{c}{0} & & & & \ddots & \\
 & & & & & & 0 & & \ddots
\end{array}
$$

First step

The proof is an induction

1. $F_1 = E_{\|T\|} \oplus E_{-\|T\|}$
2. $\pi_1^{\pm}$ $\perp$-projection on $E_{\pm\|T\|}$ $\Pi_1 = \pi_1^+ + \pi_1^-$ projection on F_1;
3. $m_1 = \dim E_{\|T\|}$, $m_2 = \dim E_{-\|T\|}$ et $d_1 = m_1 + m_2 \geq 1$;
4. $e_1, \ldots, e_{m_1}$ ONB of $E_{\|T\|}$ & $e_{m_1+1}, \ldots, e_{d_1}$ ONB of $E_{-\|T\|}$
   $e_1, \ldots, e_{d_1}$ ONB of F_1;
5. $\lambda_1 = \cdots = \lambda_{m_1} = \|T\|$, $\lambda_{m_1+1} = \cdots = \lambda_{d_1} = -\|T\|$.

$$T = \underbrace{\Pi_1 T \Pi_1}_{\sum_{k=1}^{d_1} \lambda_k \langle x, e_k\rangle e_k} + \underbrace{(I - \Pi_1) T (I - \Pi_1)}_{=T_1}$$

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First step

The proof is an induction

1. (i) $F_1 = E_{\|T\|} \oplus E_{-\|T\|}$
2. (ii) $\pi_1^{\pm}$ $\perp$-projection on $E_{\pm\|T\|}$ $\Pi_1 = \pi_1^+ + \pi_1^-$ projection on F_1;
3. (iii) $m_1 = \dim E_{\|T\|}$, $m_2 = \dim E_{-\|T\|}$ et $d_1 = m_1 + m_2 \geq 1$;
4. (iv) $e_1, \ldots, e_{m_1}$ ONB of $E_{\|T\|}$ & $e_{m_1+1}, \ldots, e_{d_1}$ ONB of $E_{-\|T\|}$
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\begin{array}{c|ccc||ccccc}
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 & & e_1,\ldots,e_{n_1} & & e_{n_1+1},\ldots,e_{n_2} & & & & \\
\hline
e_1 & \|T\| & & 0 & & & & & \\
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$$

Second step

$T_1 = (I - \Pi_1)T(I - \Pi_1)$ self-adjoint, compact, $\|T_1\| \leq \|T\|$.

Observation

$\|T_1\| < \|T\|$

Proof: If not, $\exists x$, $\|x\| = 1$, $T_1 x = \|T\|x$ (or $T_1 x = -\|T\|x$)
— $(I - \Pi_1)x = x$ otherwise $\|(I - \pi_1)x\| < \|x\|$ and then

$$\|T\| = \|T_1 x\| = \|(I - \Pi_1)T(I - \Pi_1)x\| \leq \|T\|\|(I - \Pi_1)x\| < \|T\|$$

— so $\|T\|x = T_1 x = (I - \Pi_1)T(I - \Pi_1)x = (I - \Pi_1)Tx = Tx - \Pi_1 Tx$

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Second step

$$(I-\Pi_1)x = x \ \& \ Tx = \underbrace{\|T\|x}_{\in \mathrm{Im}\,(I-\Pi_1)} + \Pi_1 Tx$$

– thus

$$\|Tx\|^2 = \|T\|^2\|x\|^2 + \|\Pi_1 Tx\|^2$$

$\Pi_1 Tx = 0$ $Tx = \|T\|x$ *i.e.* $x \in E_{\|T\|}$ a contradiction

Second step

$(I - \Pi_1)x = x$ & $Tx = \underbrace{\|T\|x}_{\in \mathrm{Im}\,(I - \Pi_1)} + \Pi_1 Tx$

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Induction

$$T = \sum_{k=1}^{n-1} \Pi_k T \Pi_k + \underbrace{\left(I - \sum_{k=1}^{n-1} \Pi_k\right) T \left(I - \sum_{k=1}^{n-1} \Pi_k\right)}_{=T_{n-1}}$$

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2. $\pi_n^{\pm}$ $\perp$-projection on $E_{\pm\|T_{n-1}\|}$ $\Pi_n = \pi_n^+ + \pi_n^-$ projection on F_n;
3. $m_{2n-1} = \dim E_{\|T_n\|}$, $m_{2n} = \dim E_{-\|T_n\|}$ et $d_n = m_{2n-1} + m_{2n} \geq 1$;
4. $e_{d_{n-1}+1}, \ldots, e_{d_{n-1}+m_{2n-1}}$ ONB of $E_{\|T_n\|}$ & $e_{d_{n-1}+m_{2n-1}+1}, \ldots, e_{d_n}$ ONB of $E_{-\|T_n\|}$ $e_{d_{n-1}+1}, \ldots, e_{d_n}$ ONB of F_1;
5. $\lambda_{d_{n-1}+1} = \cdots = \lambda_{d_{n-1}+m_{2n-1}} = \|T_{n-1}\|$, $\lambda_{d_{n-1}+m_{2n-1}+1} = \cdots = \lambda_{d_n} = -\|T_{n-1}\|$.

$$T = \sum_{k=1}^{n} \Pi_k T \Pi_k + \underbrace{(I - \sum_{k=1}^{n} \Pi_k) T (I - \sum_{k=1}^{n} \Pi_k)}$$

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Conclusion

We want to show that $T = \sum_{k=1}^{+\infty} \Pi_k T \Pi_k$ that is $T_n \to 0$.

We have $\|T_n\| < \|T_{n-1}\|$ (λ_k decreases).
Write $D_n = d_1 + \cdots + d_n$ and note that $|\lambda_{D_n}| = \cdots = |\lambda_{D_{n+1}-1}| = \|T_n\|$
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That's all!

Thank you for your attention!

http://www.u-bordeaux.fr/~pjaming/enseignement/M1.html