Spectral Theorem for compact self-adjoint operators

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http://www.u-bordeaux.fr/˜ pjaming/enseignement/M1.html

Master Mathématiques et Applications
M1 Mathématiques fondamentales & M1 Analyse, EDP, probabilités
Lecture : Introduction to spectral analysis

Preliminary warning

This video is a complement to the lecture notes available at

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1. Statement of the Spectral Theorem
2. An example

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Spectral Theorem for compact self-adjoint operators

Aim : give an example which illustrates the spectral theorem

Theorem (Spectral Theorem)

H separable, infinite dimnesional, Hilbert space. $T : H \to H$ compact, self-adjoint operator.
$\exists\ (e_k)_{k\in\mathbb{N}}$ orthonormal basis of H ;
$\exists\ (\lambda_k)_{k\in\mathbb{N}}$ real,$\lambda_k \to 0$;

$$Tx = \sum_{k\in\mathbb{N}} \lambda_k \langle x, e_k\rangle e_k.$$

Difficulty : no characteristic polynomial to find eigenvectors
Necessary to develop strategies which will depend on the (family of) operators considered.
Here : convolutions on $L^2(\mathbb{T})$.

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The operator

Hilbert space : $H = L^2(\mathbb{T})$
$= \{f : \mathbb{R} \to \mathbb{C},\ f$ 1-periodic, $\|f\|_2^2 = \int_0^1 |f(t)|^2\,\mathrm{d}t < +\infty\}$.

Operator : $T : L^2 \to L^2$, $Tf(x) = H * f(x) = \displaystyle\int_0^1 f(t)H(x-t)\,\mathrm{d}t$ with

$$H(t) = \begin{cases} 0 & \text{si } t \in (-1/2, 0) \\ 1 & \text{si } t \in (0, 1/2) \\ 1/2 & \text{if } t = 0 \text{ or } t = 1/2 \end{cases}.$$

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T is self-adjoint

$$\langle Tf, g\rangle = \int_0^1 Tf(x)\overline{g(x)}\,dx = \int_0^1\int_0^1 f(t)H(x-t)\,dt\overline{g(x)}\,dx$$

Fubini

$$= \int_0^1\int_0^1 f(t)H(x-t)\overline{g(x)}\,dx\,dt = \int_0^1 f(t)\left(\int_0^1 H(x-t)\overline{g(x)}\,dx\right)dt$$

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We are allowed to use Fubini :Cauchy-Schwarz

$$\int_0^1\int_0^1 |f(t)H(x-t)\overline{g(x)}|\,dx\,dt \leq$$

$$\int_0^1 |f(t)|\left(\int_0^1 |H(x-t)|^2\,dt\right)^{1/2}\|g\|_2\,dt \leq \int_0^1 |f(t)|\|H\|_2\|g\|_2\,dt < +\infty.$$

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$$Tf(x) = \int_0^1 K(x,t)f(t)\,\mathrm{d}t$$

with $K(x,t) = H(x-t)$.

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$= \|H\|_2^2 < +\infty$.

T is Hilbert-Schmidt thus compact.

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Theorem (Arzéla-Ascoli)

K compact metric and $X \subset \mathcal{C}(K)$ is compact $\Leftrightarrow$

1. *X is pointwise bounded : $\forall x \in K, \exists M_x > 0$ s.t. $\forall f \in X, |f(x)| \leq M_x$.*
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$f, H \in L^2(\mathbb{T})$ thus $f * H \in \mathcal{C}([0,1])$.

– $[0,1]$ is compact ;

– if $\|f\|_2 \leq 1$ then (Cauchy-Schwarz) $|H * f(x)| = \left|\int_0^1 H(x-t)f(t)\,\mathrm{d}t\right| \leq \left(\int_0^1 |H(x-t)|^2\,\mathrm{d}t\right)^{1/2} \|f\|_2 \leq \|H\|_2$(Cauchy-Schwarz) ;

$T(B)$ is pointwise bounded ;

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– if $\|f\|_2 \leq 1$ then $H * f(x) = \int_0^1 \tau_x \check{H}(t) f(t)\,\mathrm{d}t$ where $\check{H}(s) = H(-s)$ $\tau_x\varphi(s) = \varphi(s-x)$.

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T is compact - Method 3 : Kolmogorov-Riesz

Theorem (Kolmogorov-Riesz)

$X \subset L^2(\mathbb{R})$ is compact $\Leftrightarrow$

1. *X is closed ;*
2. *X is bounded :$\exists M > 0\ \forall f \in X, \|f\|_2 \leq M$;*
3. *X is equi-integrable :$\forall \varepsilon > 0\ \exists R > 0, \forall f \in X, \int_{|x| \geq R} |f(x)|^2 \, \mathrm{d}x \leq \varepsilon$.*
4. *Translations are equi-continuous on X :$\forall \varepsilon > 0, \exists \rho > 0$ s.t. $|y| < \rho \Rightarrow \forall f \in X, \|f - \tau_y f\|_2 \leq \varepsilon$.*

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Checking these conditions is done exactly in the same way.

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The spectral decomposition of T

When speaking about convolution, Fourier can't be far !

$e_k(t) = e^{2i\pi kt} \to (e_k)_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$

$$H * e_k(x) = \int_0^1 H(t)e_k(x-t)\,\mathrm{d}t = \int_0^1 H(t)e^{2i\pi k(x-t)}\,\mathrm{d}t$$

$$= \int_0^1 H(t)e^{-2i\pi kt}\,\mathrm{d}t e^{2i\pi kx} = c_k(H)e_k(x).$$

e_k is an eigenvector of T the corresponding eigenvalue is

$$c_k(H) = \begin{cases} \dfrac{1}{2} & \text{if } k = 0 \\ 0 & \text{if } k \text{ is even, } k \neq 0 \\ \dfrac{1}{ik\pi} & \text{if } k \text{ is odd} \end{cases}.$$

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$$Tf(x) = \sum_{k\in\mathbb{Z}} c_k(H)\langle f, e_k\rangle e_k(x) = \sum_{k\in\mathbb{Z}} c_k(H)c_k(f)e_k(x)$$

$$= \frac{1}{2}c_0(f) + \sum_{p\in\mathbb{Z}} \frac{1}{i(2p+1)\pi} c_{2p+1}(f)e^{2i\pi(2p+1)x}.$$

Note that we have not really used the spectral theorem to obtain this expression as we could do with out the first sum by using directly that $c_k(H * f) = c_k(H)c_k(f)$.

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Next video : Elements of the proof.

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