Spectral Theorem for compact self-adjoint operators

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http://www.u-bordeaux.fr/˜ pjaming/enseignement/M1.html

Master Mathématiques et Applications
M1 Mathématiques fondamentales & M1 Analyse, EDP, probabilités
Lecture : Introduction to spectral analysis

Preliminary warning

This video is a complement to the lecture notes available at

http://www.u-bordeaux.fr/~pjaming/enseignement/M1.html

I also assume that you have followed previous courses in which the following notions were presented
– Self-adjoint operators on Hilbert spaces : $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every x, y
– compact operators : $\overline{T(B_H)}$ is compact (B_H unit ball of H)
or equivalently, for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset H$, there exists a subsequence of (Tx_n) that converges in H (stongly)
or equivalently, if the sequence $(x_n)_{n\in\mathbb{N}} \in H$ is weakly convergent, then $(Tx_n)_{n\in\mathbb{N}}$ is strongly convergent.

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Spectral Theorem for compact self-adjoint operators

Theorem (Spectral Theorem)

H Hilbert space. T opérator $H \to H$ compact, self-adjoint. Then

- *$\exists$ a set I finite or countable*
- *$\exists$ a decomposition of H into mutually orthogonal subspaces*

$$H = \ker T \oplus \bigoplus_{i \in I} E_i = \ker T \oplus \mathrm{range}\, T$$

where each E_i has finite dimension ;

- *$\exists$ a real sequence $(\lambda_i)_{i \in I}$ with $|\lambda_i|$ non-increasing and if I infinite, $\lambda_i \to 0$;*

s.t.

$$T = \sum_{i \in I} \lambda_i \pi_i \quad \pi_i \text{ orthogonal projection on } E_i.$$

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Spectral theorem - comments

Choose $(e_{k,i})_{k=1,\ldots,n_i}$ ($n_i = \dim E_i$) on o.n.b. of E_i,

$$\pi_i x = \sum_{k=1}^{n_i} \langle x, e_{k,i}\rangle e_{k,i}$$

thus

$$Tx = \sum_{i\in I}\sum_{k=1}^{n_i} \lambda_i \langle x, e_{k,i}\rangle e_{k,i}.$$

That is, T is diagonalisable in an orthonormal basis of eigenvectors
Attention To obtain a basis of H, we mut complete with a basis of $\ker T$ which requires $\ker T$ to be *separable* (thus H as well).

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We would then like to take, $|\lambda_k|$ decreasing to 0 and $(e_k)_{k\in\mathbb{N}}$ o.n.b. of H and

$$Tx = \sum_{k\in\mathbb{N}} \lambda_k \langle x, e_k\rangle e_k$$

To do so, rewrite the basis $\bigcup_{i\in I}(e_{k,i})_{k=1,\ldots,n_i}$ and add a basis of $\ker T$.

If $\ker T$ has finite dimension, put this basis at the beginning if its co-dimension is finite, put it at the end.

Otherwise we have to mix the 2 bases and lose the fact that $|\lambda_k|$ is non-increasing

Let us put it in practice

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Théorème spectral - comments

$e_{1,1},\dots,e_{n_1,1}$ basis of $E_1 \to \tilde e_1,\dots,\tilde e_{n_1}$
$e_{1,2},\dots,e_{n_1,2}$ basis of $E_2 \to \tilde e_{n_1+1},\dots,\tilde e_{n_1+n_2}$
either it stops at some $\tilde e_N$ (case $co-\dim\ker T<+\infty$) or gives an infinite sequence
take a basis of $\ker T (f_1,\dots,f_M)$ if $M=\dim\ker T<+\infty$or $(f_k)_{k\in\mathbb{N}}$ otherwise.

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Spectral theorem - comments

Unite both bases

If $co-\dim\ker T<+\infty$, $(e_k)_{k\geq 1}$ is $\to \tilde e_1,\ldots,\tilde e_N, f_1,\ldots$the corresponding eigenvalues are $\lambda_1,\ldots,\lambda_N,0\ldots$ are decreasing to 0

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Theorem (Spectral Theorem revisited)

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$\exists\ (e_k)_{k\in\mathbb{N}}$ orthonormal basis of H ;

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$\exists\ (e_k)_{k\in\mathbb{N}}$ orthonormal basis of H ;

$\exists\ (\lambda_k)_{k\in\mathbb{N}}$ real,$\lambda_k\to 0$;

$$Tx=\sum_{k\in\mathbb{N}}\lambda_k\langle x,e_k\rangle e_k.$$

Spectral theorem - comments

Unite both bases

If $co-\dim\ker T<+\infty$, $(e_k)_{k\geq 1}$ is $\to \tilde e_1,\ldots,\tilde e_N, f_1,\ldots$ the corresponding eigenvalues are $\lambda_1,\ldots,\lambda_N, 0\ldots$ are decreasing to 0

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Spectral theorem - Finite dimensional case

Finite dimensional case : self adjoint matrices are diagonalizable in an orthonormal basis of eigenvectors$A = A^* \Rightarrow \exists U$ unitary $(U^{-1} = U^*) \exists D$ real diagonal$A = UDU^*$.

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one can also interpret this as T having a matrix of the form

$$
\begin{array}{c|c|cccccc}
 & \ker T & \multicolumn{2}{c}{E_1} & \multicolumn{2}{c}{E_2} & \multicolumn{2}{c}{\cdots} \\
 & & \multicolumn{2}{c}{e_{1,1},\dots,e_{n_1,1}} & \multicolumn{2}{c}{e_{1,2},\dots,e_{n_2,2}} & & \\
\hline
\ker T & 0 & \multicolumn{2}{c}{0} & \multicolumn{2}{c}{0} & \multicolumn{2}{c}{\cdots} \\
\hline
e_{1,1} & & \lambda_1 & 0 & & & & \\
\vdots & 0 & \multicolumn{2}{c}{\ddots} & & & \multicolumn{2}{c}{0} \\
e_{n_1,1} & & 0 & \lambda_1 & & & & \\
e_{1,2} & & & & \lambda_2 & 0 & & \\
\vdots & 0 & & & \multicolumn{2}{c}{\ddots} & & \\
e_{n_2,2} & & & & 0 & \lambda_2 & & \\
 & & & & & & \ddots & 0 \\
 & 0 & \multicolumn{2}{c}{0} & & & \multicolumn{2}{c}{\ddots} \\
 & & & & & & 0 & \ddots
\end{array}
$$

That's all !

Thank you for your attention !

Next video : an example.

http://www.u-bordeaux.fr/~pjaming/enseignement/M1.html

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