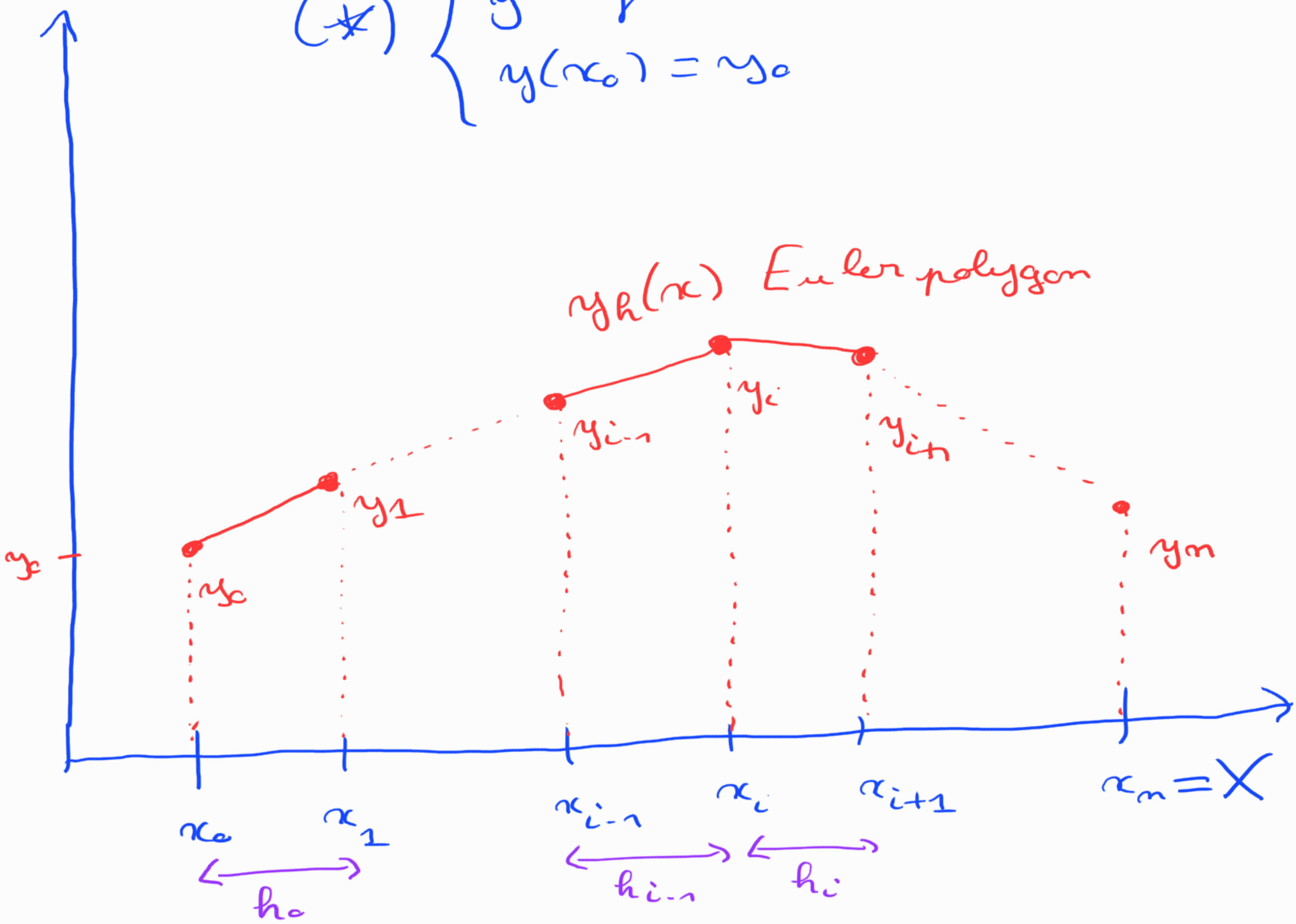
Convergence study of Euler Method

$$(*) \begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$



Main steps of the proof:

Stability

$$\bullet \begin{cases} D = \{ (x,y) \quad x_0 \le x \le X, \ |y - y_0| \le b \} \\ |f| \le A \text{ on } D \\ X - x_0 \le \dfrac{b}{A} \end{cases}$$

$$\Longrightarrow \begin{cases} \bullet \text{ the numerical solution remains in } D \\ \bullet \ |y_h(x) - y_0| \le A\,|x - x_0| \\ \bullet \ |y_h(x) - (y_0 + (x - x_0) f(x_0, y_0))| \le \varepsilon\,|x - x_0| \end{cases}$$

if $f(x,y) - f(x_0, y_0) \le \varepsilon$ on D

Stability with respect to initial condition:

$h = (h_0, \text{———}, h_{n-1})$ fixed

$\left.\begin{matrix} y_h(x) \\ z_h(x) \end{matrix}\right\} =$ Euler polygons for initial values y_0 and z_0.

$\left| \frac{\partial f}{\partial y}(x,y) \right| \le L$ in a convex region containing $(x, y_h(x))$, $(x, z_h(x))$

with $x_0 \le x \le X$

$$\Longrightarrow \boxed{|z_h(x) - y_h(x)| \le e^{L(x - x_0)} |z_0 - y_0|}$$

Convergence of numerical solution to exact sol

f is continuous, $|f|$ bounded by A

and satisfies Lipschitz condition on D

If $X - x_0 \le \frac{b}{A}$ then:

If $\max_i h_i \to 0$ then $y_h(x)$ converges to ϕ the unique sol of $(*)$

Error estimates

Suppose that in the neighborhood of the solution

$|f| \le A \qquad \left|\frac{\partial f}{\partial y}\right| \le L \qquad \left|\frac{\partial f}{\partial x}\right| \le M$

Then:

$$|y(x)-y_h(x)| \leq \frac{M+AL}{L}\left(e^{L(x-x_0)}-1\right)\max_i h_i$$

if $\max_i h_i$ is small enough.