Convergence study of the Euler method

We consider the one-dimensional differential equation :

$$\begin{aligned} y' &= f(x,y) && (1)\\ y(x_0) &= y_0. && (2)\end{aligned}$$

We are looking for a numerical approximation of $y(X)$ with $X > x_0$.

We consider also a subdivision of the interval of integration :

$$x_0, x_1, x_2, \ldots, x_{n-1}, x_n = X$$

and compute on each point x_i with $1 \leq i \leq n$ the numerical solution y_i approximation $y(x_i)$ using Euler's method :

$$y_{i+1} = y_i + (x_{i+1} - x_i) f(x_i, y_i) \tag{3}$$

For the subdivision we use the notation :

$$h_i = x_{i+1} - x_i$$

and we denote

$$h = (h_0, h_1, \ldots, h_n). \tag{4}$$

If we connect y_0 and y_1, y_1 and y_2, etc by straight lines we obtain the so-called Euler polygon defined by :

$$y_h(x) = y_i + (x - x_i) f(x_i, y_i) \text{ for } x_i \leq x \leq x_{i=1}. \tag{5}$$

Proposition 0.1. *We assume that $|f|$ is bounded by A on the domain $D = \{(x,y) \,|\, x_0 \leq x \leq X, |y - y_0| \leq b\}$, and that $X - x_0 \leq \frac{b}{A}$. Then the numerical solution given by (3) remains in D for every subdivision of the type (4) and*

$$\begin{aligned} |y_h(x) - y_0| &\leq A|x - x_0|, && (6)\\ |y_h(x) - \big(y_0 + (x - x_0) f(x_0, y_0)\big)| &\leq \epsilon |x - x_0| && (7)\end{aligned}$$

if $|f(x,y) - f(x_0,y_0) \leq \epsilon|$ on D.

Démonstration. Both inequalities are obtained by summing the relationships (3) are using triangle inequalities.

Next step is to find an estimation for the variation of the numerical solution if the initial value changes. Let z_0 be another initial condition. We compute

$$z_1 = z_0 + (x_1 - x_0) f(x_0, y_0) \tag{8}$$

and look for an estimate of $|z_1 - y_1|$. We substract (8) from the respective relationship of the type (3) to obtain :

$$z_1 - y_1 = z_0 - y_0 + (x_1 - x_0)\big(f(x_0, y_0) - f(x_1, y_1)\big)$$

We thus need an estimate for $\big(f(x_0, y_0) - f(x_1, y_1)\big)$. If we suppose that f satisfies a Lipschitz condition

$$\big(f(x_0, y_0) - f(x_1, y_1)\big) \quad \leq \quad L|z - y| \tag{9}$$

we obtain

$$|z_1 - y_1| \leq (1 + (x_1 - x_0)L)|z_0 - y_0|.$$

Proposition 0.2. *For a fixed subdivision h let $y_h(x)$ and $z_h(x)$ be the Euler polygons corresponding to the initial values y_0 and z_0 respectively. If*

$$|\frac{\partial f}{\partial y}(x, y)| \quad \leq \quad L \tag{10}$$

in a convex region containing $(x, y_h(x))$ and $(x, z_h(x))$ for all $x_0 \leq x \leq X$, then

$$|z_h(x) - y_h(x)| \quad \leq \quad e^{L(x - x_0)}|z_0 - y_0| \tag{11}$$

Démonstration. Inequality (10) implies

$$\big(f(x_0, y_0) - f(x_1, y_1)\big) \quad \leq \quad L|z - y|$$

which implies

$$|z_1 - y_1| \leq e^{L(x_1 - x_0)}|z_0 - y_0|$$

Then we repeat the same argument for $z_2 - y_2$, $z_3 - y_3$ etc to obtain the final inequality.

Now we want to handle the convergence of the numerical solution to the exact solution of the differential equation.

Proposition 0.3. *We suppose that f is continuous and $|f|$ is bounded by A and satisfies a Lipschitz condition on the domain D. If $X - x_0 \leq \dfrac{b}{A}$ then we have*

- *If $\max_i h_i \to 0$ the Euler polygon $y_h(x)$ converge uniformly to a continuous fonction ϕ.*
- *ϕ is continuously differentiable and is the unique solution of the differential equation (1) on $x_0 \leq x \leq X$*

Démonstration. Let us consider $\epsilon > 0$. Since f is uniformly continuous on the compact set D there exists a $\delta > 0$ such that

$$|u_1 - u_2| \leq \delta \text{ and } |v_1 - v_2| \leq A\,\delta$$

implies

$$|f(u_1, v_1) - f(u_2, v_2)| \quad \leq \quad \epsilon \tag{12}$$

We suppose now that the subdivision h satisfies

$$\max_i h_i \quad \leq \quad \delta. \tag{13}$$

We first study the effect of adding new mesh points. In a first step, we consider a subdivision $h(1)$ which is obtained by adding new points only to the first subinterval. It follows from (7) (applied to this first subinterval) that for the new refined solution $y_{h(1)}(x_1)$ we have the estimate

$$|y_{h(1)}(x_1) - y_h(x_1)| \leq \epsilon |x_1 - x_0|.$$

Since the subdivisions h and $h(1)$ are identical on $x_1 \leq x \leq X$ we can apply Proposition 0.2 to obtain

$$|y_{h(1)}(x) - y_h(x)| \leq e^{L(x-x_1)}(x_1 - x_0)\epsilon \text{ for } x_1 \leq x \leq X.$$

We next add further points to the subinterval (x_1, x_2) and denote the new subdivision by $h(2)$. In the same way as above this leads to

$$|y_{h(2)}(x_2) - y_{h(1)}(x_2)| \leq \epsilon |x_2 - x_1|$$

and

$$|y_{h(2)}(x) - y_{h(1)}(x)| \leq e^{L(x-x_2)}(x_2 - x_1)\epsilon \text{ for } x_2 \leq x \leq X.$$

If we denote by $\hat{h}$ the final refinement, we obtain for $x_i \leq x \leq x_{i+1}$

$$\begin{aligned} |y_{\hat{h}}(x) - y_h(x)| &\leq \epsilon \Big(e^{L(x-x_1)}(x_1 - x_0) + \ldots + e^{L(x-x_i)}(x_i - x_{i-1}) \Big) + \epsilon (x - x_i) \\ &\leq \epsilon \int_{x_0}^{x} e^{L(x-s)} ds = \frac{\epsilon}{L} \Big(e^{L(x-x_0)} - 1 \Big) \end{aligned}$$

If now we have two different subdivisions h and $\tilde{h}$, which both satisfy (13), we introduce a third subdivision $\hat{h}$ which is a refinement of both subdivisions, and apply (14) twice. We then obtain from (14) by the triangle inequality

$$|y_h(x) - y_{\tilde{h}}(x)| \leq 2 \frac{\epsilon}{L} \Big(e^{L(x-x-0)} - 1 \Big).$$

For $\epsilon > 0$ small enough, this becomes arbitrary small and shows the convergence of the Euler polygons to a continuous function $\phi(x)$.

Now we define

$$\epsilon(\delta) = sup \left\{ |f(u_1, v_1) - f(u_2, v_2)| \, ; \, |u_1 - u_2| \leq \delta, |v_1 - v_2| \leq A\delta, (u_i, v_i) \in D \right\}.$$

If x belongs to the subdivision h then we obtain from (7) (replace (x_0, y_0) by $(x, y_h(x))$ and x by $x + \delta$)

$$|y_h(x + \delta) - y_h(x) - \delta f(x, y_h(x))| \quad \leq \quad \epsilon(\delta)\,\delta.$$

Taking the limit $\max\limits_i h_i \to 0$ we get

$$|\phi(x + \delta) - \phi(x) - \delta\, f(x, \phi(x))| \quad \leq \quad \epsilon(\delta)\,\delta.$$

Since $\epsilon(\delta) \to 0$ for $\delta \to 0$ this proves the differentiability of $\phi(x)$ and $\phi'(x) = f(x, \phi(x))$.

Now let $\psi(x)$ be a second solution of the differential equation and suppose that the subdivision h satisfies (13). We then denote by $y_h^{(i)}(x)$ the Euler polygon to the initial value $(x_i, \psi(x_i))$, which is defined for $x_i \le x \le X$. It follows from

$$\psi(x) = \psi(x_i) + \int_{x_i}^{x} f(s, \psi(s))\, ds$$

and (12) that

$$|\psi(x) - y_h^{(i)}(x)| \le \epsilon |x - x_i| \text{ for } x_i \le x \le x_{i+1}.$$

Using Proposition 0.2 we deduce in the same way as above that

$$|\psi(x) - y_h(x)| \quad \le \quad \frac{\epsilon}{L}\Big(e^{L(x-x_0)} - 1\Big).$$

Taking the limit $\max_i h_i \to 0$ and $\epsilon \to 0$ we obtain $|\psi(x) - \phi(x)| \le 0$, which proves uniqueness.

Proposition 0.4. *Suppose that in a neighborhood of the solution*

$$|f| \le A\ ,\ |\frac{\partial f}{\partial y}| \le L\ ;\ |\frac{\partial f}{\partial x}| \le M.$$

We then have the following error estimate for the Euler polygons :

$$|y(x) - y_h(x)| \quad \le \quad \frac{M + AL}{L}\Big(e^{L(x-x_0)} - 1\Big) \max_i h_i$$

provided that $\max_i h_i$ *is sufficiently small.*

Démonstration. For $|u_1 - u_2| \le \max_i h_i$ and $|v_1 - v_2| \le A \max_i h_i$ we obtain, due to the differentiability of f, the estimate

$$|f(u_1, v_1) - f(u_2, v_2)| \le (M + AL)\max_i h_i.$$

When we insert this amount for ϵ into (14), we obtain the stated result.