# Kepler's Problem

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May 11, 2008

For a nice non-mathematical introduction to Kepler take a look at [3]. It's a good read.

An elementary modern introduction to classical mechanics is [4]. This book has a clean discussion of the 2-body problem and transfer orbits and gives you the tools to solve a number of interesting problems. Plan your own Grand Tour!

## 1 A Bit of History

Eudoxus who was born around 400 BC constructed a solar system model involving some 27 spheres. Aristotle, wanting more precision, added another 29 spheres, bringing the total to 56! Later, around 125 AD, Hipparchus and Ptolemy worked out an epicycle theory. They probably realized that their descriptions were simply encodings of observational data and a system of extrapolation and not a physical theory. An analogous idea is to approximate any period process by a trigonometric polynomial, that is, by a partial sum of a Fourier series. It works, but is hardly a physical theory. It is precisely the fact that the epicycle description is not a physical theory, and not its geocentrism which is its major drawback. The accuracy was only tolerable, say lunar eclipses within an hour, but could be improved, at the cost of greater complexity, by adding additional epicycles.

By some mysterious process the Ptolemaic vision became official church doctrine in medieval Europe - the circles and constant angular velocities in the model became articles of faith! This says a lot more about the nature of bureaucrazy than it does about the solar system! It's particularly frightening if we remember that our ancestors in medieval Europe (and earlier in Classical Greece) had precisely the same mental equipment as we do.

Copernicus (1473-1543) published his heliocentric theory of the solar system in 1543. His theory used epicycles as did the geocentric theory of Ptolemy but it was simpler (though not simple) to use and required far fewer epicycles. The heliocentric theory was not really in any better accord with observation than the current version of the geocentric Ptolemy theory but it treated the motions of all the planets in the same way. Ptolemy, as you might expect, had to treat the inner and outer planets differently. Both systems are capable of refinement by introducing additional epicycles (and complexity) but that would diminish one of the main reasons to adopt heliocentrism. The discrepancies between the initial heliocentric theory and observation were so large that agreement with observation was not a compelling reason to adopt heliocentrism in place of geocentrism. Astronomers such as Tycho Brahe (1546-1601) were not inclined to adopt it though people who computed tables found it more convenient.

Copernicus' achievement was not just the adoption of heliocentrism but an emphasis on simplicity (a relative term here) and his opposition to entrenched authority.

Johannes Kepler (1571-1630) worked as an assistant to Tycho Brahe in his observatory. After 14 years of observing Mars he formulated his three laws of motion for the planets (stated below) and introduced a version of heliocentrism based on the ellipse rather than the circle (and totally devoid of epicycles). This was an impressive achievement. I can not resist quoting Hubbard and West [6]:

> Even knowing that these laws are true, and using a calculator and a telescope, it is not clear what you should observe in order to confirm them. One cannot be but struck at the amazing genius it must have taken for a person, without a telescope or any accurate means of measurement, before the invention of analytic geometry, and without any way to determine the distances of any celestial bodies, to take the results of 14 years of observations and come up with such laws. If you consider that Tycho's observatory was on an island between Denmark and Sweden, which must be, next to Ithaca, the cloudiest place in the world, the records must have been pretty spotty in the extreme.

Of course Kepler was not concerned just with the shape of the orbit but also with the time element, that is, the problem of predicting the position at a given time. Kepler's equation gives the time in terms of the position. As you might expect, Kepler's problem is the problem of solving for the position in terms of the time. Isaac Newton (1642–1727), among many other things, proved that Kepler's equation cannot be solved in algebraic functions. Numerous people have devised schemes for finding approximate solutions.

In 1738 Daniell Bernoulli (1700-1782) published a paper concerning the oscillation of heavy chains. The power series for the Bessel function $J_0$ occurs in this paper, probably for the first time. The Bessel functions of arbitrary integral order first occured in the work of Leonhard Euler (1707-1783) on vibrations of membranes in 1764. The Bessel functions of small integral order occur in the approximate solution to Kepler's problem found by Joseph-Louis Lagrange (1736-1813) around 1770. Circa 1816 Friedrich Wilhelm Bessel (1784-1846) discovered the exact version of Lagrange's solution to Kepler's problem (as an infinite series). We will derive this solution below. About the same time F. Carlini studied the asymptotic properties of Bessel functions in order to investigate the convergence of the Lagrange–Bessel solution to Kepler's problem. Around 1824 Bessel investigated in detail the properties of the Bessel functions and laid the foundations for an area of study which has grown a great deal since then.

A great deal of modern analysis, in particular analytic function theory, grew out of the investigations of Lagrange, Carlini, Bessel and others on Kepler's problem, To some extent Augustin Cauchy (1759–1857) was led to create a large part of modern complex analysis because of his study of convergence of a series solution of Kepler's problem, [1]. The other main creators of complex analysis and analytic function theory were of course Weierstrass (1815–1897) and Riemann (1826–1866), but we do not discuss them here.

## 2 The Two Body Problem

The 2-body problem deals with two point masses moving in Euclidean 3-space in accord with the differential equations:

$$\begin{aligned} m_1 \frac{d^2\vec{u}_1}{dt^2} &= Gm_1m_2 \frac{\vec{u}_2 - \vec{u}_1}{\|\vec{u}_2 - \vec{u}_1\|^3} \\ m_2 \frac{d^2\vec{u}_2}{dt^2} &= Gm_1m_2 \frac{\vec{u}_1 - \vec{u}_2}{\|\vec{u}_2 - \vec{u}_1\|^3} \end{aligned}$$

where $G$ is Newton's *universal constant of gravitation*. With breathless courage Newton declared $G$ to be constant throughout the universe. This claim of course makes it possible to compute $G$ from near Earth, or even laboratory, measurements. It is known that

$$G = 6.670 \times 10^{-8} \text{ dyne-cm}^2/\text{gm}^2$$

though I confess I have very little feeling for the somewhat mysterious units.

If we add the two equations together we deduce that the center of mass

$$\vec{R}_0 = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$$

moves along a straight line with constant velocity. This observation yields 6 first integrals for the system above and so allows us to reduce the order to $12 - 6 = 6$. We can achieve the reduction explicitly by referring the motion to the center of mass. This amounts to using a new inerial frame.

Let $\vec{r}_1 = \vec{u}_1 - \vec{R}_0$ and $\vec{r}_2 = \vec{u}_2 - \vec{R}_0$. A quick calculation shows

$$\begin{aligned} \frac{d^2\vec{r}_1}{dt^2} &= -\mu_1 \frac{\vec{r}_1}{\|\vec{r}_1\|^3}, \qquad \mu_1 = \frac{Gm_2^3}{(m_1+m_2)^2} \\ \frac{d^2\vec{r}_2}{dt^2} &= -\mu_2 \frac{\vec{r}_2}{\|\vec{r}_2\|^3}, \qquad \mu_2 = \frac{Gm_1^3}{(m_1+m_2)^2}. \end{aligned}$$

Here $\mu_1$ and $\mu_2$ are known as the *gravitational parameters*. At first sight it appears we still have a system of order 12, but in fact, the new vector equations are decoupled. Thus we can solve one of them, an equation of order 6 and then, of course, use $m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$ to instantly write down the solution of the other. In other words, the order is effectively reduced to 6. Since our new frame is unaccelerated the laws of physics remain the same in the new coordinates.

Remarkably, there is also a non-inertial change of coordinates which simplifies the equations and effectively reduces the order to 6. The idea is to place an observer on $m_1$ and to view the motion from there. Thus we let $\vec{r}_3 = \vec{r}_2 - \vec{r}_1$. A quick calculation shows

$$\frac{d^2\vec{r}_3}{dt^2} = -\mu_3 \frac{\vec{r}_3}{\|\vec{r}_3\|^3}, \qquad \mu_3 = G(m_1+m_2).$$

In this case we tend to forget that we are actually using an accelerated frame of reference and we even have the temerity to view the observer as being at rest. In particular, if we speak of a fixed plane, we actually mean a plane moving in such a way that it appears fixed to our non-inertial observer.

Whichever viewpoint we choose the equation of motion is of order 6 and has the form

$$\frac{d^2\vec{r}}{dt^2} = -\mu \frac{\vec{r}}{\|\vec{r}\|^3}.$$

It will be convenient to introduce the notation: give a vector $\vec{a}$ then $a$ designates the magnitude $\|\vec{a}\|$. This notation is regrettably susceptible to the error known as the "departed arrows," but is useful nonetheless. In this new notation we have

$$\frac{d^2\vec{r}}{dt^2} = -\mu \frac{\vec{r}}{r^3}.$$

This equation is sometimes referred to, jokingly, as the *one body problem* or, more properly, as the *inverse square central force equation.*

We first note

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = -\mu \frac{\vec{r} \times \vec{r}}{r^3} = \vec{0}$$

and therefore

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{0}.$$

Thus

$$\vec{h} = \vec{r} \times \frac{d\vec{r}}{dt}$$

is a constant vector. It is called the *specific angular momentum* vector. Technically we could use the constancy of $\vec{h}$ to reduce the order by another 3. Conservation of energy provides another first integral, allowing reduction to order 2. Eliminating the time then yields a first order equation of motion, which we can solve. The solution yields the orbits, but without the time dependence, since we eliminated it. Solving explicitly for the time dependence is Kepler's problem. We will arrive at this point soon, but not quite by the path described.

If $\vec{h} = \vec{0}$ then $\vec{r}$ and $\frac{d\vec{r}}{dt}$ are parallel, corresponding to motion directly away from the observer, or directly towards. The first choice is pretty dull, and the second is exciting, at least to the observer, since it leads to a collision in finite time.

If $\vec{h} \neq \vec{0}$ then $\vec{h} = \vec{r} \times \frac{d\vec{r}}{dt}$ implies that $\vec{r}$ and $\frac{d\vec{r}}{dt}$ are always perpendicular to $\vec{h}$. Thus the motion takes place in a "fixed" plane through the observer and perpendicular to $\vec{h}$. This plane is called the *orbital plane*.

Handedness or orientation is actually a bit subtle in the mathematical version of Euclidean 3-space since we can not actually place our right hand (or, left for that matter) in it. Fortunately it doesn't much matter, and it's safe to pretend we know what we mean. With this proviso we choose a right-handed rectangular Cartesian coordinate system with fundamental basis $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$ and $\vec{\mathrm{k}}$ such that $\vec{\mathrm{i}}$ and $\vec{\mathrm{j}}$ lie in the orbital plane. Then

$$\vec{r} = r\cos(\theta)\vec{\mathrm{i}} + r\sin(\theta)\vec{\mathrm{j}}$$

where $r$ and $\theta$ are functions of time, and are simply polar coordinates in the orbital plane. Then

$$\frac{d\vec{r}}{dt} = \left(\frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}\right)\vec{\mathrm{i}} + \left(\frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}\right)\vec{\mathrm{j}}$$

and therefore

$$\vec{h} = \vec{r} \times \frac{d\vec{r}}{dt} = r^2\frac{d\theta}{dt}\vec{\mathrm{k}}$$

and so

$$\frac{1}{2}h = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{dS}{dt}$$

where $S$ is the area swept out in the orbital plane by the position vector $\vec{r}$. Thus we have obtained Kepler's second law

> (K2): The position vector traces out equal areas in equal times in the orbital plane, that is, the *areal velocity* is constant.

Now, since $\vec{h}$ is constant

$$\frac{d^2\vec{r}}{dt^2} \times \vec{h} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt} \times \vec{h}\right)$$

From the equation of motion we have

$$\begin{aligned}
\frac{d^2\vec{r}}{dt^2} \times \vec{h} &= -\frac{\mu}{r^3}\vec{r} \times \vec{h} \\
&= -\frac{\mu}{r^3}\vec{r} \times \left(\vec{r} \times \frac{d\vec{r}}{dt}\right) \\
&= -\frac{\mu}{r^3}\left(\left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right)\vec{r} - (\vec{r} \cdot \vec{r})\frac{d\vec{r}}{dt}\right) \\
&= -\frac{\mu}{r^3}\left(\left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right)\vec{r} - r^2\frac{d\vec{r}}{dt}\right)
\end{aligned}$$

Now

$$\begin{aligned}
\vec{r} \cdot \frac{d\vec{r}}{dt} &= \frac{1}{2}\frac{d}{dt}(\vec{r} \cdot \vec{r}) \\
&= \frac{1}{2}\frac{d}{dt}r^2 \\
&= r\frac{dr}{dt}
\end{aligned}$$

Thus

$$\frac{d^2\vec{r}}{dt^2} \times \vec{h} = \mu\left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right) = \mu\frac{d}{dt}\left(\frac{1}{r}\vec{r}\right).$$

By equating the two expressions for $\frac{d^2\vec{r}}{dt^2} \times \vec{h}$ and integrating we obtain that there is a constant vector $\vec{e}$, called the *eccentricity vector*, such that

$$\frac{d\vec{r}}{dt} \times \vec{h} = \mu\left(\frac{1}{r}\vec{r} + \vec{e}\right).$$

From this equation it follows that $\frac{1}{r}\vec{r}+\vec{e}$ is perpendicular to $\vec{h}$ and so lies in the orbital plane. But $\vec{r}$ lies in the orbital plane and so we conclude the vector $\vec{e}$ lies in the orbital plane. Thus we may assume that we have chosen our coordinates so that

$$\vec{e} = e\vec{i}.$$

Thus

$$\vec{r} \cdot \left(\frac{d\vec{r}}{dt} \times \vec{h}\right) = \mu \left(\frac{1}{r}\vec{r} \cdot \vec{r} + e\vec{r} \cdot \vec{i}\right) = \mu r\left(1 + e\cos(\theta)\right)$$

where, as usual, the polar angle $\theta$ is the angle between $\vec{r}$ and $\vec{i}$. But

$$\vec{r} \cdot \left(\frac{d\vec{r}}{dt} \times \vec{h}\right) = \vec{h} \cdot \left(\frac{d\vec{r}}{dt} \times \vec{r}\right) = \vec{h} \cdot \vec{h} = h^2.$$

Thus we have

$$r = \frac{h^2/\mu}{1+e\cos(\theta)}$$

which is the polar equation of a conic section with parameter $h^2/\mu$ and eccentricity e. Moreover $\theta$ is the true anomaly and since $\theta$ is measured from $\vec{i}$ and $e \geq 0$ we conclude that the eccentricity vector $\vec{e}$ points towards periapsis. At any rate we have established what is essentially Kepler's first law:

> (K1): The motion takes place along a conic section with the observer at a focus.

Recall $e = 0$ for a circle, $0 < e < 1$ for an ellipse, $e = 1$ for a parabola, and $e > 1$ for a hyperbola. Kepler's laws actually arose from his study of the orbit of Mars and so were formulated only for an elliptical orbit. As we have seen the first two laws actually hold also for parabolic and hyperbolic orbits.

Consider now the case of an elliptical orbit $(0 \leq e < 1)$. As we saw above periapsis (minimum distance $r_{\text{min}}$) occurs when $\theta = 0$. By symmetry apoapsis (maximum distance $r_{\text{max}}$) occurs when $\theta = \pi$. Thus

$$r_{\text{min}} = \frac{h^2/\mu}{1+e} \text{ and } r_{\text{max}} = \frac{h^2/\mu}{1-e}.$$

If $a$ is the semi-major axis then $2a = r_{\text{min}} + r_{\text{max}}$. Thus

$$a = \frac{h^2}{(1-e^2)\mu}$$

and so

$$(1-e^2)^{1/2} = h\mu^{-1/2}a^{-1/2}.$$

The area of the ellipse is given by

$$S = \pi ab = \pi(1 - e^2)^{1/2}$$

and the areal velocity we have already seen is

$$\frac{h}{2}.$$

Thus the period of the motion is given by

$$P = \frac{S}{h/2} = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

which yields Kepler's third law:

> (K3): In the case of elliptical motion the square of the period is proportional to the cube of the mean distance $a$.

In the 2-body problem the orbit is usually described by 6 parameters.

1. A scale factor, usually the semi-major axis $a$ (for elliptical orbits) or the parameter $p$ (in general) otherwise known as the semi-latus rectum.
2. The eccentricity e.
3. The inclination of the orbit relative to some reference plane through the observer's position.
4. The longitude of the ascending node, that is, where the orbit goes from below the reference plane to above it.
5. The argument of periapsis, that is, the angle between the ascending node and periapsis.
6. One of, the time of periapsis or the true anomaly at epoch, that is, the true anomaly at the time of observation.

For all the wonderful details check [2], an excellent textbook. You may also find [7] very useful. An interesting, but unfortunately non-mathematical, description of how Newton's law of gravity was used to discover Neptune is given in [5]. It is great light reading.

Note the orbit may be determined in a number of ways from observtional data:

- Gibbs' method: from 3 successive position vectors.
- Gauss' method: from 2 position vectors and time of flight between them.
- Laplace's method: from 3 angular positions (distances unknown) and times.

# 3 Bessel Functions

For everything you could possibly want to known about Bessel functions check the treatise [8]. This book should be in your private library!

For each $z \in \mathbb{C}$

$$w \longrightarrow \mathrm{e}^{z(w-\frac{1}{w})/2}$$

is an analytic function in $\mathbb{C} \sim \{0\}$. Hence we have a Laurent series

$$\mathrm{e}^{z(w-\frac{1}{w})/2} = \sum_{n=-\infty}^{\infty} \mathrm{J}_n(z) w^n, \quad w \neq 0.$$

The coefficient $\mathrm{J}_n$ is called the Bessel function of order $n$. By replacing $1/w$ by $-w$ we see easily

$$\mathrm{J}_{-n}(z) = (-1)^n \mathrm{J}_n(z).$$

If $r > 0$ and $\gamma(t) = r\mathrm{e}^{\mathrm{i}t}$, $0 \leq t \leq 2\pi$, then the formula for the coefficients of the Laurent series yields

$$\mathrm{J}_n(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma w^{-n-1} \mathrm{e}^{z(w-1/w)/2}\, dw.$$

It follows that $\mathrm{J}_n$ is an entire function. Taking $r = 1$ we obtain

$$\mathrm{J}_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}z\sin t - \mathrm{i}nt}\, dt.$$

If we replace $\gamma(t)$ by $\gamma_1(t) = \mathrm{e}^{-\mathrm{i}t}$ we obtain instead

$$\mathrm{J}_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}z\sin t + \mathrm{i}nt}\, dt.$$

By adding these two expressions (and dividing by 2) we obtain

$$\mathrm{J}_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \cos(z \sin t - nt)\, dt.$$

If we replace $t$ by $2\pi - t$ on the interval $[\pi, 2\pi]$ we finally obtain

$$\mathrm{J}_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin t - nt)\, dt.$$

# 4 Kepler's Problem

One of the tools Kepler introduced to describe an elliptical orbit is the circumscribed circle. If we consider a point $Q$ on the ellipse with true anomaly $\theta$ and project it onto the circle perpendicularly to the major axis of the ellipse we obtain a point with central angle $E$ called the *eccentric anomaly* of $Q$. We have a number of relations:

$$\begin{aligned}
\sin(\theta) &= \frac{(1-e^2)^{1/2}\sin(E)}{1-e\cos(E)} \\
\sin(E) &= \frac{(1-e^2)^{1/2}\sin(\theta)}{1+e\cos(\theta)} \\
r\cos(\theta) &= a(\cos(E)-e) \\
r\sin(\theta) &= a(1-e^2)^{1/2}\sin(E) \\
r &= a(1-e\cos(E)) \\
\cos(\theta) &= \frac{\cos(E)-e}{1-e\cos(E)} \\
\tan(\theta/2) &= \left(\frac{1+e}{1-e}\right)^{1/2}\tan(E/2).
\end{aligned}$$

I am sure with effort you could find many more relationships!

The area swept out by the radius vector from periapsis is

$$\frac{1}{2}\int_0^\theta r^2\,\mathrm{d}\theta = \frac{1}{2}a^2(1-e^2)^{1/2}(E-e\sin(E)) = \frac{ab}{2}(E-e\sin(E)).$$

This is a bit of a calculation that you may enjoy doing. You can do it geometrically as Kepler must have done, or you can make a suitable change of variable in the integral.

The quantity

$$M = E - \sin(E)$$

is called the *mean anomaly*. Let $T$ be the time of periapsis and let $P$ be the period of the elliptical orbit. Since the areal velocity is constant we have the area swept out at time $t$ is

$$\frac{t-T}{P}\pi ab = \frac{t-T}{P}\pi a^2(1-e^2)^{1/2}.$$

Comparing this result with the calculation above we obtain

$$\frac{t-T}{P} = E - \sin(E), \text{ or } M = \frac{t-T}{P}.$$

Thus to predict the position ($E$) at a given time ($t$) we have to solve the equation $M = E - \sin(E)$ for $E$. This is Kepler's problem. Since $0 \le e < 1$ in the elliptical case we see that $M$ is a strictly increasing function of $E$ and so in principle we can solve for $E$ in terms of $M$.

Suppose we have found the solution $E = g(M)$ to Kepler's equation. Then an easy calculation shows

$$g(M+2\pi) = g(M) + 2\pi$$

and therefore $g(M) - M$ is periodic with period $2\pi$. Note also that it is odd. Thus we can consider a Fourier sine expansion,

$$g(M) - M = \sum_{n=1}^{\infty} a_n \sin(nM)$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} (g(M) - M) \sin(nM) \, \mathrm{d}M.$$

Making the change of variable $M = E - \sin(E)$ and integrating by parts, etc., we obtain

$$a_n = \frac{2}{n\pi} \int_0^{\pi} \cos(nE - ne\sin(E)) \, \mathrm{d}E = \frac{2}{n} J_n(ne).$$

Thus we obtain Bessel's solution to Kepler's problem

$$E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin(nM).$$

Such a series is called a Kapteyn series (see [8]) and converges fairly rapidly if $0 \leq e < 1$. One can also show

$$\frac{r}{a} = 1 + \frac{1}{2}e^2 - \sum_{n=1}^{\infty} \frac{2e}{n} J_n'(ne) \cos(nM),$$

another Kapteyn series. Prior to Bessel's solution there were a number of approximate methods including graphical ones. Note that Newton's iteration for approximating roots works quite well for estimating solutions to Kepler's equation. Newton's method may even have been invented for that purpose. Perhaps you can find out!

# 5 Problems

Here's a couple of "simple" problems. You can find more problems of this sort n [4].

**Problem 1.** Explorer VI on Feb, 3, 1960 had a perigee distance of 6,627.6 km and an apogee distance of 48,201.0 km. Compute the period.

Note the mass of Explorer VI was much smaller than the mass of the Earth so we can take $\mu = GM$ where $M$ is the mass of the Earth. Then

$$\mu = 3.986032 \times 10^5 \text{ km}^3/\text{sec}^2.$$

The actual period was 45,166.2 sec and your answer should be pretty close. Note due to various pertubations the period was decreasing by about 2.2644 sec per revolution.

**Problem 2.** Given that Mars has a satellite (Phobos) of small mass in a circular orbit of radius 9,330 km with period 27,540 sec, use only the major semi-axis and the period of the Earth satellite Explorer VI given above to estimate the mass of Mars in terms of the mass of the Earth.

Note the accepted value is about 0.108. Your answer should be within 3 %.

# References


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